7.1 Riemann Integral

If $I = [a,b]$ is a closed bounded interval, then a partition of I is a set

$P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_n = b.$$

We define $I_1 = [x_0, x_1]$,

$I_2 = [x_1, x_2], \ldots I_n = [x_{n-1}, x_n]$

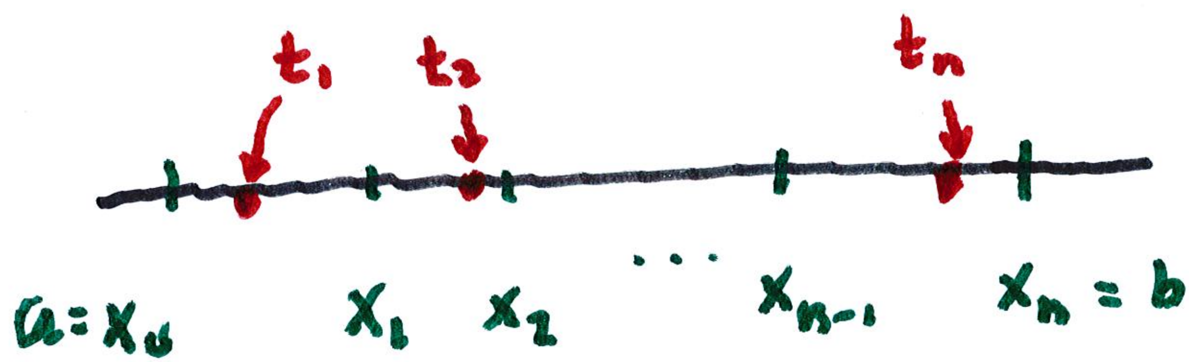
We define the norm of P by

$$\|P\| = \max\left\{ x_1 - x_0,\ x_2 - x_1,\ \ldots\ x_n - x_{n-1} \right\}$$

If t_i has been selected from $I_i = [x_{i-1}, x_i]$, the points are called tags of the intervals I_i.

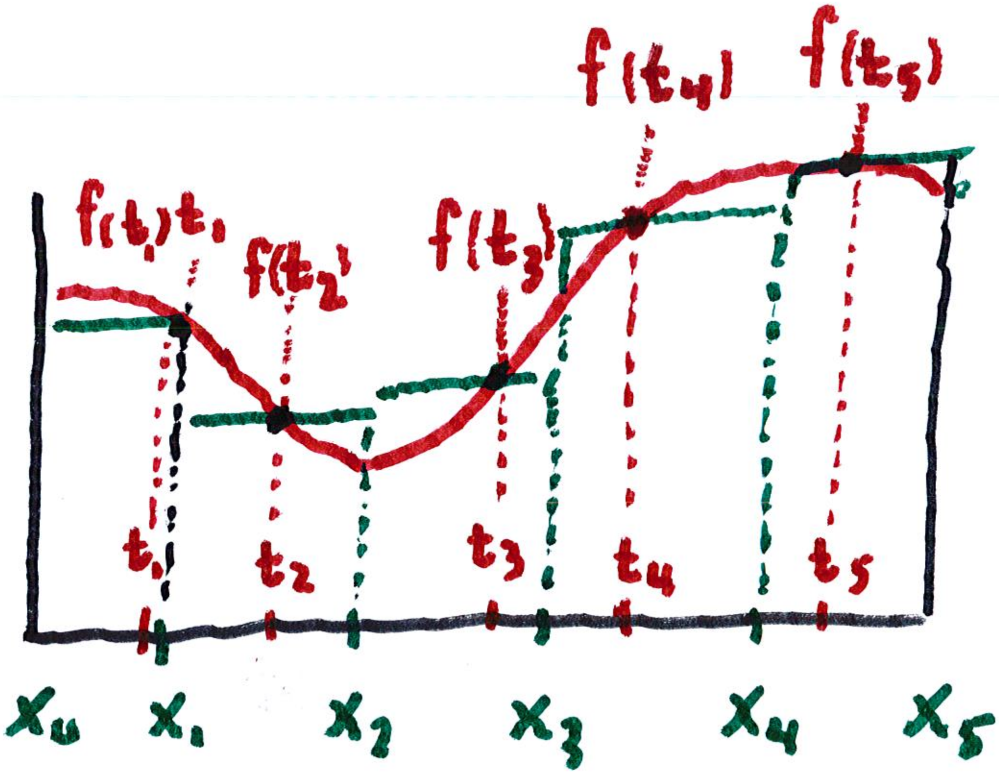
$$\dot{P} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^{n}$$

is called a **tagged partition** of I.



Given a tagged partition, we define the Riemann sum of $f : [a, b] \longrightarrow \mathbb{R}$ by

$$S(f, \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$



Area of i-th rectangle

$= f(t_i)(x_i - x_{i-1})$

We define the Riemann integral of a function f on $[a, b]$

Def'n. A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there is a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $\|\dot{P}\| < \delta_\varepsilon$, then

$$|S(f; \dot{P}) - L| < \varepsilon.$$

The set of all integrable functions is denoted by $R[a, b]$.

(One can say $L =$ limit of the Riemann sum $S(f, \dot{P})$ as $\|\dot{P}\| \to 0$.)

We write $L = \int_a^b f$ or

$$L = \int_a^b f(x)\,dx$$

Thm. If $f \in R[a,b]$, then the value of the integral is uniquely determined.

Pf. Suppose L' and L'' satisfy the definition.

Let $\varepsilon > 0$. Then there exists

$\delta'_{\varepsilon/2}$ such that if $\dot{P}$ is

any tagged partition with

$\|\dot{P}\| < \delta'_{\varepsilon/2}$, then

$$\left| S(f; \dot{P}) - L' \right| < \frac{\varepsilon}{2}$$

Also, there exists $\delta''_{\varepsilon/2}$ such that if $\dot{P}$ is any tagged partition with

$\| \dot{P} \| < \frac{\varepsilon}{2}$, then

$$\left| S(f; \dot{P}) - L'' \right| < \frac{\varepsilon}{2}.$$

Set $\delta = \min \left\{ \delta'_{\varepsilon/2} , \delta''_{\varepsilon/2} \right\}$

and let $\dot{P}$ be any tagged partition

with $\| \dot{P} \| < \delta_\varepsilon$, then

$$\left| S(f;\dot{P}) - L' \right| < \frac{\varepsilon}{2}$$ and

$$\left| S(f;\dot{P}) - L'' \right| < \frac{\varepsilon}{2}.$$

Hence the Triangle Inequality implies:

$$\left| L' - L'' \right| = \left| L' - S(f;\dot{P}) + S(f;\dot{P}) - L'' \right|$$

$$\leq \left| L' - S(f;\dot{P}) \right| + \left| S(f;\dot{P}) - L'' \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

it follows that $L' = L''$.

We now give some examples.

Ex 1. Every constant function

on $[a, b]$ is in $R[a, b]$.

Let $f(x) = k$, for $x \in [a, b]$,

and let $\dot{P} = \{x_0, \dots x_N ; t_i\}_{i=1}^{N}$

$$S(f; P) = \sum_{i=1}^{n} k(x_i - x_{i-1})$$

$$= k(x_n - x_0) = k(b-a).$$

For any $\varepsilon > 0$, set $\delta = 1$,

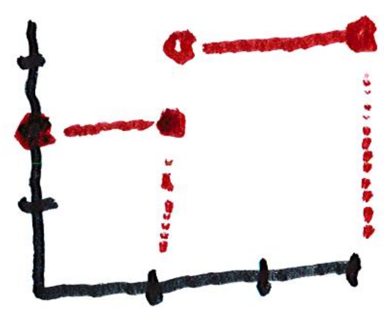
so that if $\|\dot{P}\| < \delta$, then

$$\left| S(f, P) - k(b-a) \right| = 0 < \varepsilon.$$

Ex. 2. Let $g(x) = \begin{cases} 2, & \text{if } 0 \le x \le 1 \\ 3, & \text{if } 1 < x \le 3 \end{cases}$

Let $\dot{P}$ be any tagged partition of $[0,3]$, where $\|\dot{P}\| < \delta$.

Let j be the largest integer such that $x_j \leq 1$. Note that this implies that $x_{j+1} > 1$.



We compute $S(g;\dot{P})$.

$$S(g;P) = 2\sum_{i=1}^{j}(x_i - x_{i-1})$$

$$+ \pi(2,3)\,(x_{j+1} - x_j)$$

$$+ 3\,(x_N - x_{j+1}),$$

(where $\pi(2,3) = 2$ or 3)

$$\leq 2x_j + \pi(2,3)\,(x_{j+1} - x_j)$$

$$+ 3\,(3 - x_{j+1}).$$

We now estimate $S(g;\dot{P})$.

using the fact that

$|x_i - x_{i-1}| < \delta$ for all $i = 1, \dots, N$.

$$S(g;P) \le 2 + 3\delta + 9 - 3$$

(using $x_{j+1} > 1$)

$$= 8 + 3\delta.$$

$$S(g;\dot{P}) \ge 2(1-\delta) + 2\delta$$

$$+ 3(1) = 8$$

This implies $|S(g;\dot{P}) - 8| < 3\delta$

If we set $\delta = \frac{\varepsilon}{3}$.

$\Rightarrow$ if $\|\dot{P}\| < \delta$, then

$|S(g;\dot{P}) - 8| < \varepsilon$.

Ex. 3. Compute $\int_0^1 x^2\,dx$.

Let $\dot{Q}$ be the partition

$\{x_0, x_1, \ldots, x_N\}$ with the

tag defined by $t_i = q_i = \dfrac{x_i + x_{i-1}}{2}$.

Then $h(x) \equiv x^2$ satisfies

$$h(q_i)(x_i - x_{i-1}) = \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$$

$$= \frac{1}{2}(x_i^2 - x_{i-1}^2).$$

This sum telescapes:

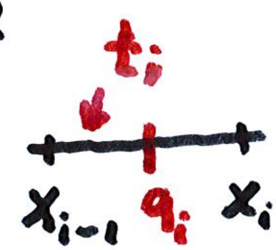
$$S(h; \dot{Q}) = \sum_{i=1}^{n} \frac{1}{2}(x_i^2 - x_{i-1}^2)$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}.$$

Now let $\dot{P}$ be an arbitrary partition of $[0,1]$ with $\|\dot{P}\| < \delta$.

We use q_i = midpoint of I_i.

Note that $|t_i - q_i| < \frac{\delta}{2}$



Using the Triangle Inequality

$$\left| S(h;\dot{P}) - S(h;Q) \right|$$

$$= \left| \sum_{i=1}^{n} t_i (x_i - x_{i-1}) - \sum_{i=1}^{n} q_i (x_i - x_{i-1}) \right|$$

$$\leq \sum_{i=1}^{n} |t_i - q_i| (x_i - x_{i-1})$$

$$< \frac{\delta}{2} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\delta}{2}(1 - 0) = \frac{\delta}{2}.$$

Since $S(h;\dot{Q}) = \frac{1}{2}$, we conclude

that if $\|\dot{P}\| < \delta$, then

$$\left| S(h;\dot{P}) - \tfrac{1}{2} \right| < \frac{\delta}{2}.$$

Setting $\delta = 2\varepsilon$, we obtain that if $\|\dot{P}\| < \delta = 2\varepsilon$, then

$$\left| S(h; P) - \frac{1}{2} \right| < \varepsilon.$$