7.1 Riemann Integral cont'd

Thm. If $f \in \mathcal{R}[a,b]$, then f is bounded on $[a,b]$.

Pf. Assume f is unbounded on $[a,b]$ and that $\int_a^b f = L$.

Then there is $\delta > 0$ so that if $\dot{\mathcal{P}}$ is any tagged partition with

$\|\dot{P}\| < \delta$, then $|S(f; \dot{P}) - L| < 1$,

which implies that

$$
\begin{aligned}
|S(f; \dot{P})| &= |S(f, \dot{P}) - L + L| \\
&< |S(f; P) - L| + |L| \\
&< 1 + |L| \qquad (1)
\end{aligned}
$$

Now suppose $Q = \{[x_{i-1}, x_i]\}_{i=1}^{n}$

is a partition of $[a, b]$ with

$\|Q\| < \delta$. Since f is unbounded there exists a subinterval in Q, say $[x_{k-1}, x_k]$ on which $|f|$ is not bounded. Now we pick tags for Q, so $t_i = x_i$ for $i \neq k$ and we pick $t_k \in [x_{k-1}, x_k]$ so

$$|f(t_k)(x_k - x_{k-1})| > L + 1 + \left| \sum_{i \neq k} f(t_i)(x_i - x_{i-1}) \right|$$

The Backward Triangle Inequality

implies

$$|S(f;\dot{Q})| \geq |f(t_k)(x_k - x_{k-1})|$$

$$- \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1 ,$$

which contradicts (1). Hence

$f \in R[a,b] \Rightarrow |f|$ is bounded.

7.2 Riemann Integrable Functions

Thm. Cauchy Criterion.

A function : $[a,b] \to \mathbb{R}$ is in

$R[a,b]$ if and only if

for every $\varepsilon > 0$ there is $\eta_\varepsilon > 0$

such that if $\dot{P}$ and $\dot{Q}$ are any

tagged ~~functions~~ partitions of $[a,b]$

with $\|\dot{P}\| < \eta_\varepsilon$ and $\|\dot{Q}\| < \eta_\varepsilon$, then $\left| S(f;\dot{P}) - S(f;Q) \right| < \varepsilon.$

Proof: $(\Rightarrow)$ If $f \in \mathcal{R}[a,b]$ and

$L = \int_a^b f$, let $\eta_\varepsilon = \delta_{\varepsilon/2}$ be

such that if $\dot{P}$, $\dot{Q}$ are tagged

partitions such that

$\|\dot{P}\| < \eta_\varepsilon$ and $\|\dot{Q}\| < \eta$, then

$$\left| S(f;\dot{P}) - L \right| < \frac{\varepsilon}{2} \text{ and }$$

$$\left| S(f;\dot{Q}) - L \right| < \frac{\varepsilon}{2}.$$

Hence

$$|S(f;\dot{P}) - S(f;\dot{Q})|$$

$$\leq |S(f;\dot{P}) - L + L - S(f;\dot{Q})|$$

$$\leq |S(f;\dot{P}) - L| + |L - S(f;\dot{Q})|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$) For each $n \in N$, let $\delta_n > 0$ such that if $\dot{P}$ and $\dot{Q}$ are tagged partitions with norms $< \delta_n$, then

$$\left| S(f; \dot{P}) - S(f; \dot{Q}) \right| < \frac{1}{n}.$$

We can assume that $\delta_n > \delta_{n+1}$ for $n \in N$; otherwise, replace δ_n by $\delta_n' = \min\{\delta_1, \dots \delta_n\}$

For each $n \in N$, let $\dot{P}_n$ be a tagged partition with $\|\dot{P}_n\| < \delta_n$.

Clearly, if $m > n$, then both

$\dot{P}_m$ and $\dot{P}_n$ have norms $< \delta_n$,

so that

$$|S(f;\dot{P}_n) - S(f;\dot{P}_m)| < \frac{1}{n} \tag{2}$$

for $m > n$.

Hence, the sequence

$(S(f;\dot{P}_m))_{m=1}^{\infty}$ is a Cauchy

sequence in $\mathbb{R}$ and we let

$$A = \lim_m S(f; \dot{P}_m).$$

Passing to the limit in (2) as $m \to \infty$, we have

$$|S(f; \dot{P}_n) - A| < \frac{1}{n} \quad \text{for all } n \in N$$

To see that A is the Riemann integral of f, given $\varepsilon > 0$, let $K \in N$ satisfy $K > 2/\varepsilon$.

If $\dot{Q}$ is any tagged partition with $\|\dot{Q}\| < \delta_K$, then

$$|S(f;\dot{Q})-A| \le |S(f;Q) - S(f;\dot{P}_K)| + |S(f;\dot{P}_K) - A|$$

$$\le \frac{1}{K} + \frac{1}{K} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f \in R[a,b]$ with integral A.

7.1.3 Squeeze Thm..

Let $f: [a,b] \to \mathbb{R}$. Then $f \in R[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions $\alpha_\varepsilon(x)$ and $\omega_\varepsilon(x)$ in $R[a,b]$ with

(3) $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$, all $x \in [a,b]$

and such that

(4) $$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon .$$

Proof: $(\Rightarrow)$ Set $\alpha_\varepsilon = \omega_\varepsilon = f$ for all $\varepsilon > 0$.

$(\Leftarrow)$ Let $\varepsilon > 0$. Since α_ε and ω_ε belong to $R[a,b]$, there exists $\delta_\varepsilon > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta_\varepsilon$, then

$$\left| S(\alpha_\varepsilon ; \dot{P}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon$$

and

$$\left| S(\omega_\varepsilon ; \dot{P}) - \int_a^b \omega_\varepsilon \right| < \varepsilon .$$

It follows that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon ; \dot{P})$$

and

$$S(\omega_\varepsilon ; \dot{P}) < \int_a^b \omega_\varepsilon + \varepsilon .$$

By (3) we have

$$S(\alpha_\varepsilon ; \dot{P}) \le S(f; \dot{P}) \le S(\omega_\varepsilon ; \dot{P}),$$

and hence

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{P}) < \int_a^b \omega_\varepsilon + \varepsilon$$

If $\dot{Q}$ is another tagged partition with $\|\dot{Q}\| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{Q}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If we subtract these inequalities

and use (4), we conclude that

$$\left| S(f;\dot{P}) - S(f;\dot{Q}) \right|$$

$$< \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon$$

$$= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the

Cauchy Criterion implies that

$f \in R[a,b]$