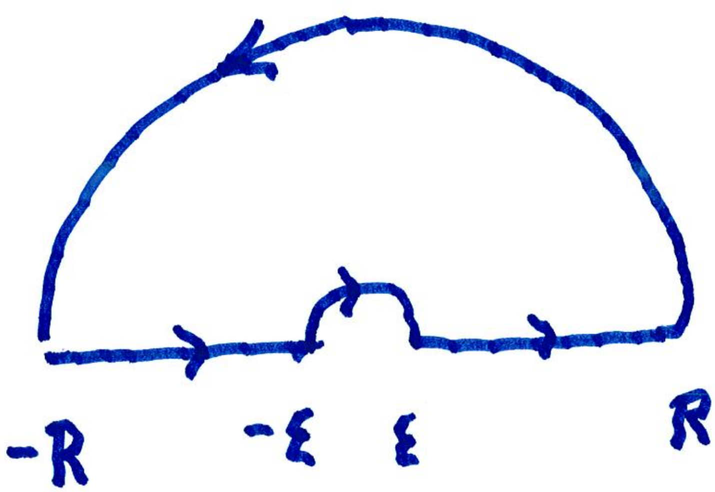
Compute $$\int_0^\infty \frac{1-\cos x}{x^2}\,dx \qquad (1)$$

We first compute $$\int_{-\infty}^{\infty} \frac{1-e^{iz}}{z^2}\,dz$$

Consider



$\gamma_R(t) = Re^{it}$

By Cauchy's Thm:

$$\underbrace{\int_{-R}^{-\varepsilon} \frac{1-e^{iz}}{z^2}\,dz}_{I_1} - \underbrace{\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz}_{I_2} \qquad (2)$$

$$+ \underbrace{\int_{\varepsilon}^{R} \frac{1-e^{iz}}{z^2}\,dz}_{I_3} + \underbrace{\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,dz}_{I_4} = 0$$

Note $|e^{x+iy}| \le 1$ if $y \ge 0$

$$|I_4| \leq \int_0^{\pi} \frac{2}{R^2} R \, d\theta = \frac{2\pi}{R} \to 0$$

$\therefore$ As $R \to \infty$

$$I_1 \to \int_{-\infty}^{-\varepsilon} \frac{1-e^{iz}}{z^2} \, dz$$

and

$$I_3 \to \int_{\varepsilon}^{\infty} \frac{1-e^{iz}}{z^2} \, dz$$

For I_2, note that

$$\frac{1-e^{iz}}{z^2} = \frac{-i}{z} + E(z),$$

where $E(z)$ is bounded

as $z \to 0$.

Using $z(t) = \varepsilon e^{it}$ for $0 \le t \le \pi$,

$$\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz = \int_0^\pi \left(\frac{-i}{\varepsilon e^{it}} + E(\varepsilon e^{it})\right) i\varepsilon e^{it}\,dt$$

The part involving $E \to 0$

as $\varepsilon \to 0$

The first part equals π

Since the integral I_2

also has a minus sign, {in (1)}

We conclude that

$-I_2 \to -\pi$ as $\varepsilon \to 0$.

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} = \pi.$

Thm. Holomorphic Functions defined by an Integral.

Suppose $F(z,s)$ is defined for $z \in \Omega$ and s in $[a,b]$, where Ω is an open set in $\mathbb{C}$. Suppose also that

(i) F is holomorphic in z for each s

(iii) F is continuous on $\Omega \times [a,b]$.

Then $f(z)$, defined by

$$f(z) = \int_a^b F(z,s)\,ds$$

is holomorphic in Ω.

Pf. Let $n \geq 1$, and set z_k

$= a + k\delta$, where $\delta = \frac{b-a}{n}$

To see this, recall that a continuous function on a compact set is uniformly continous, so if $\varepsilon > 0$, there is a $\sigma > 0$ such that

$$\sup_{z \in \overline{D}} \left| F(z, s_1') - F(z, s_2') \right| < \varepsilon$$

if $|s_1' - s_2'| < \sigma$.

Choose a disc D so that $\overline{D} \subset \Omega$. Then consider the sum

$$f_n(z) = \sum_{k=1}^{n} F(z, \zeta_k)\,\delta$$

Then f_n is holomorphic in all of Ω by (i). Also $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f

Then, if $n > \frac{1}{\sigma}$ and $z \in \overline{D}$,

we have

$$|f_n(z) - f(z)|$$

$$= \left| \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} F(z, s_k) - F(z, s)\, ds \right|$$

$$\leq \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} |F(z, s_k) - F(z, s)|\, ds$$

$$< \sum_{k=1}^{n} \frac{\varepsilon(b-a)}{n} = \varepsilon(b-a).$$

Then, since f_n converges uniformly to f, it follows that f is also holomorphic on D. Since D is arbitrary in Ω, it follows that f is holomorphic on Ω.

Riemann Removable Singularities.

Suppose that f is holomorphic in the punctured disc $D_r(z_0) - z_0$. and suppose that $|f(z)| \le M$ for all $z \in D_r(z_0) - z_0$. Then there is a function F that is analytic on $D_r(z_0)$ so that $F = f$ on $D_r(z_0) - z_0$.

Proof: Define $g(z) = f(z)(z-z_0)^2$ if $z \neq z_0$ and set $g(z_0) = 0$.

Clearly g is holomorphic in $D_r(z_0) - z_0$. Also, g is holomorphic at z_0 with $g'(z_0) = 0$. By the power series expansion theorem

we can write

$$(z-z_0)^2 f(z) = g(z) = \sum_{n=2}^{\infty} a_n (z-z_0)^n$$

(since g vanishes to order at z_0 at least 2) Dividing

by $(z-z_0)^2$, we see that

$$F(z) = \sum_{n=0}^{\infty} a_{n+2} (z-z_0)^n .$$

Zeroes and Poles

A point singularity of a holomorphic function f is a complex number z_0 such that f is defined in a neighborhood of z_0 but not at z_0 itself.

We just proved that if $|f(z)| < M$ for z near z_0, then z_0 is a removable singularity.

We will say a function f has a <u>pole at z_0</u> if

$$\lim_{z \to z_0} |p(z)| = \infty.$$

For such a function $f(z) \neq 0$ in a punctured disc. If we set

$g(z) = \dfrac{1}{f(z)}$, then g satisfies

$\lim_{z \to z_0} g(z) = 0$, since $\lim_{z \to z_0} f(z) = \infty$.

It follows that g has a removable singularity, so that we can think of $g(z)$ as a holomorphic function in a neighborhood D of z_0.

Also, g vanishes to order at least $m \geq 1$.

Hence, we can write

$$f(z) = (z-z_0)^{-m} \frac{1}{h(z)}, \text{ where}$$

h is holomorphic and nonzero near z_0. If we write

$$\frac{1}{h(z)} = a_0 + a_1 \overset{(z-z_0)}{z} + \cdots + a_m z^m + \cdots$$

~~then~~ $\cancel{f(z) = a_0 z^{-m} + a_1 z^1}$ then

$$f(z) = a_0 z^{-m} + a_1 z^{-(m-1)} + a_2 z^{-(m-2)} + \cdots$$

$$= \quad + a_{m-1} \frac{1}{z} + \sum_{k=0}^{\infty} a_{m+k} z^k$$