Some Notation

$$D'_r(z_0) = \{ z ;\ 0 < |z - z_0| < r \}$$

If f is holomorphic in Ω,

we write $f \in A(\Omega)$.

Suppose that $f \in A(D'_r(z_0))$

and that $\lim\limits_{z \to z_0} |f(z)| = \infty$.

By shrinking r if necessary,

we can assume that $|f(z)| > 1$, for all $z \in D_r'(z_0)$. If we set

$g(z) = \dfrac{1}{f(z)}$, then $|g(z)| \cancel{>}< 1$, $z \in D_r'(z_0)$

Hence g has a removable singularity at z_0. Since $\lim\limits_{z \to z_0} g(z) = 0$

it follows that $g(z_0) = 0$.

$\therefore$ There is an integer $m \geq 1$ so that g vanishes to order m at zero.

and that

that $g(z) = (z - z_0)^m H(z)$,

where $H(z)$ is holomorphic

and nonzero on $D_r(z_0)$. If we

set $H(z) = \frac{1}{h(z)}$, then

$$f(z) = (z - z_0)^{-m} h(z), \quad z \in D_r'(z_0).$$

Since $h(z) = a_0 + a_1 z + a_2 z^2 + \ldots$

converges in $D_r(z_0)$, we conclude

that

$$f(z) = a_0 (z - z_0)^{-m} + a_1 (z - z_0)^{-m+1}$$

$$+ \dots + a_{m-1} (z - z_0)^{-1}$$

$$+ \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n.$$

The sum of the first m terms on the left are called the Principal Part $P_r(z)$. Thus, we see that f is the sum of m singular terms in P_r plus a function $E(z)$ that is

holomorphic in $D_r(z_0)$.

we see f has a pole at z_0.

There are 3 kinds of

isolated singularities for a

function f in $D_r(z_0)$:

(i) f has a removable singularity

$$(|f| < M \text{ in } D'_r(z_0))$$

OR

(ii) f has a pole at z_0

$$\left(\lim_{z \to z_0} |f(z)| = \infty\right)$$

and hence f can be written

$$f(z) = P_r(z) + E(z).$$

OR

(iii) f has neither a removable

singularity nor a pole at z_0

We say f has an essential singularity

Casorati - Weierstrass Thm.

If f has an essential singularity at z_0

then for any $r > 0$,

the image of $D_r'(z_0)$ under f

is dense in $\mathbb{C}$.

pf. Suppose the theorem is NOT true. Then there is $w \in \mathbb{C}$ and a $\delta > 0$ so that

$$|f(z) - w| \geq \delta, \quad \text{for all } z \in D_r'(z_0)$$

Then $$\left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\delta}, \quad z \in D_r'(z_0).$$

and so, $\dfrac{1}{f(z) - w}$ has a removable singularity at z_0.

Hence $$\frac{1}{f(z) - w} = h(z) = \sum_{n=N}^{\infty} c_n (z - z_0)^n$$

where $c_N \neq 0$. If $N=0$, f has a removable sing which is a contradiction.

If $N>0$, ~~Hence~~ $f(z)$ has a pole of order N at z_0, which is a contradiction.

Thus f has the density property in each disc $D'_r(z_0)$.

Def'n. A meromorphic function consists of the following:

(i) a sequence (possibly empty, finite, or infinite) $S = \{z_n\}$, with no limit point in Ω

(ii) a sequence of positive numbers $\{r_n\}$ such that each pair of discs

$$D_{r_{n_1}} \cap D_{r_{n_2}} = \emptyset,$$

and $\bar{D}_{r_n} \subset \Omega$ for every n,

and

(iii) a function $F \in A(\Omega - S)$

and (iv) a sequence of

functions $P_n(z)$.

of the form

$$P_n(z) = \sum_{k=1}^{m_n} c_k^n (z - z_n)^{-k}$$

such that in $D'_{r_n}(z_n)$

$F(z) - P_n(z)$ extends

to a function in $A(D_{r_n}(z_n))$

Suppose that f has a pole at z_0. Thus,

$$f(z) = \sum_{k=-n}^{k=\infty} a_k (z-z_0)^k, \quad z \in D'_r(z_0)$$

We can write

$$E(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

which is holomorphic in $D_r(z_0)$.

If C is a circle about z_0 of radius R, then Cauchy's Thm implies

$$\int_{C_R} E(z)\,dz = 0.$$

If $k < -1$, then

$$\left(\frac{a_k}{1+k} (z-z_0)^{1+k} \right)' = a_k (z-z_0)^k$$

so $a_k (z-z_0)^k$ has a primitive

$$\Rightarrow \int_{C_R} a_k (z-z_0)^k = 0 .$$

On the other hand, when

$k=-1$, we have

$$\int_{C_R} a_{-1}(z-z_0)^{-1}\,dz$$

$$= \int_{C_R} \frac{a_{-1}\,dz}{z-z_0} = 2\pi i \cdot a_{-1}$$

using $z(t) = z_0 + Re^{it}$,

$0 \le t \le 2\pi$

Thus if f has a pole at z_0, then the coefficient a_{-1} of $(z-z_0)^{-1}$ is called the residue of f,

$$\operatorname{res}_{z_0} f = a_{-1}.$$

Two important facts

(1) If f has a pole of order 1, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

(2) If f has a pole of order n, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z - z_0)^n f(z).$$

In fact,

$$(z - z_0)^n f(z) = a_{-n} + \dots + a_{-1} (z - z_0)^{n-1} + G(z)(z - z_0)^n$$

Since a_{-1} is the coefficient

of $(z-z_0)^{m-1}$, it follows that

the above formula holds.

Thm. (The Residue formula)

Suppose that f is holomorphic

in an open set containg a

circle C and its interior,

except for poles at

the points $z_1, \dots, z_N$ inside

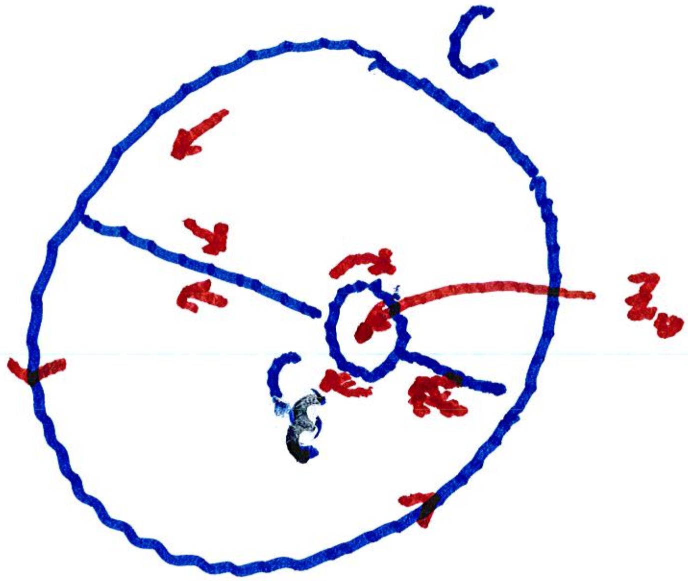
C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

Consider first the case when

$N=1$.

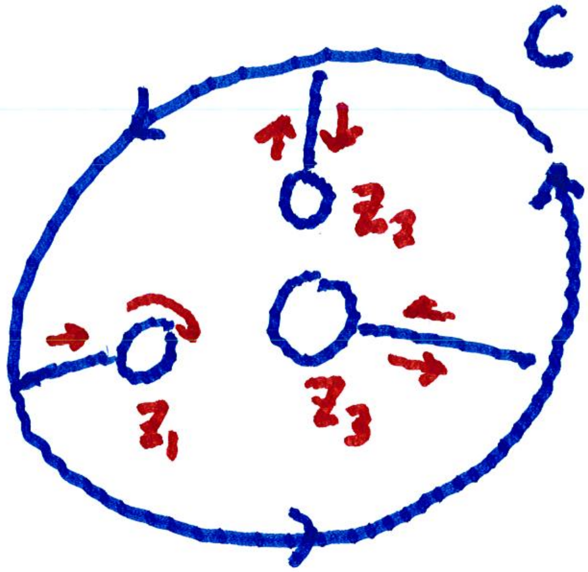
$$\int_C f(z)\,dz - \int_{C_\varepsilon(z_0)} f(z)\,dz = 0.$$



We conclude that

$$\int_{C_\varepsilon} f(z)\,dz = \operatorname{Res}_{z_0} f$$

In the general case



$$\therefore \int_C f(z)\,dz = \sum_{k=1}^{N} \operatorname{res}_{z_k} f \cdot 2\pi i$$