Four kinds of definite integrals

1. $$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

Set $z = e^{i\theta}$

$$\rightarrow \cos\theta = \frac{z + z^{-1}}{2} \qquad \sin\theta = \frac{z - z^{-1}}{2i}$$

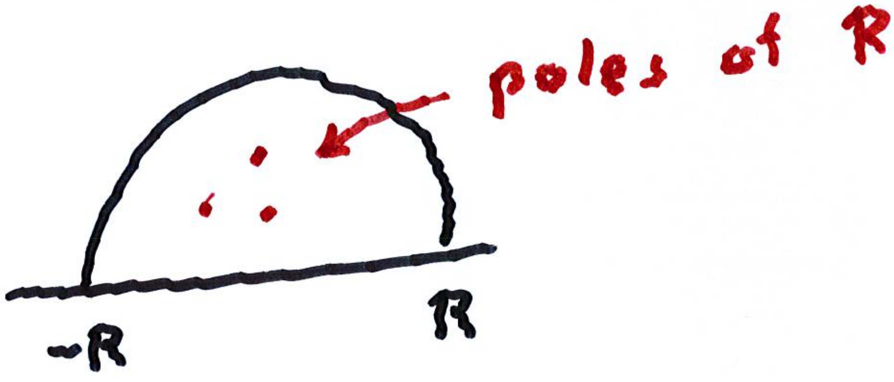
and $d\theta = \dfrac{dz}{iz}$

2. $\displaystyle\int_{-\infty}^{\infty} R(x)\,dx$, where $R(x) = \dfrac{P(x)}{Q(x)}$

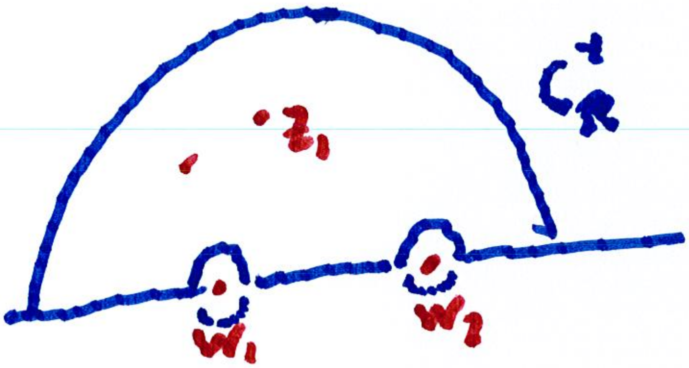
and $\deg P \le \deg Q - 2$

and there are no poles of

$R(x)$ on the real line.



$\exists$ poles on real line



$$\int_{\gamma_R} R(x)\,dx = \frac{1}{2}\, 2\pi i \sum \mathrm{Res}\; R(z)$$

$$+\; 2\pi i \sum \mathrm{Res}_{z_j} R(z)$$

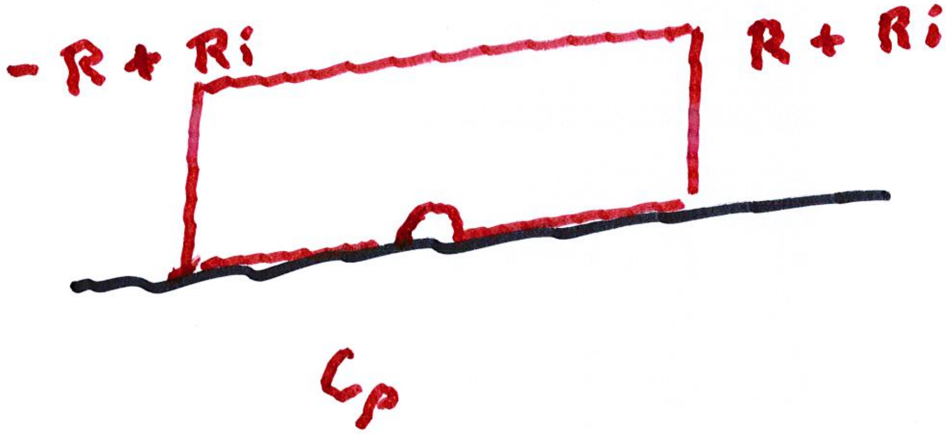
Need $\deg P \leq \deg Q - 2$

4. $\int R(x)\sin x$ or $\int R(x)(1-\cos x)\,dx$

$\deg P \le \deg Q - 1$

Convert $\sin x$ to e^{ix}, etc.

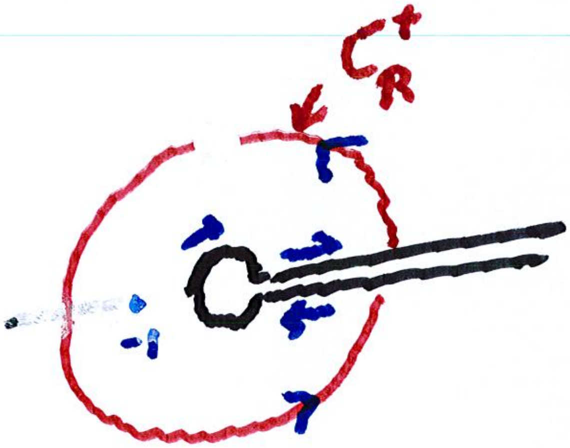
Use this path



$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y}$$

5. Integrate along a

branch cut.



For z^b, use that

$z^b R(z) = C_b z^b R(z)$

lower edge

upper edge.

$z^{-a} = e^{-a \log z}$

$$\int_0^\infty \frac{x^{-a}}{1+x}$$

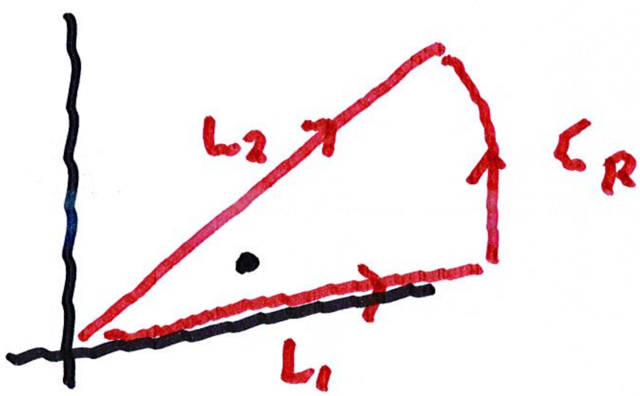
$\log z$ $e^{-a(\log x + i\theta)}$

$\theta = 2\pi \Rightarrow e^{-a \log z} e^{-ai2\pi}$

Just 1 more definite integral.

$$\int_0^\infty \frac{dx}{x^n+1} \qquad , \; n \geq 2$$

Consider the path



Let the angle be $\frac{2\pi i}{n}$

Then x^n+1 has a simple pole at $z = e^{\frac{\pi i}{n}}$

The Residue formula implies

that

$$\int_{L_1} - \int_{L_2} + \int_{C_R} = 2\pi i \, \mathrm{Res}_{e^{i\pi/n}} f(z)$$

where $f(z) = \dfrac{1}{1+z^n}$

If $g(z)$ has a simple pole at z_0, then the residue of $\dfrac{1}{g(z)}$ is $= \dfrac{1}{g'(z_0)}$.

Setting $g(z) = z^n + 1$, then

$$\operatorname{Res}_{e^{\pi i/n}}\left(\frac{1}{z^n+1}\right) = \frac{1}{n\left(e^{\pi i/n}\right)^{n-1}}$$

$$= \frac{1}{n} e^{-\frac{\pi i}{n}}.$$

In the usual way,

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{1+z^n} \to 0, \quad \text{since } n > 1.$$

With $I = \int_0^\infty \frac{dz}{z^n+1}$, then

$$I = \frac{2\pi i}{1 - e^{2\pi i/n}} \cdot e^{-\frac{\pi i}{n}}$$

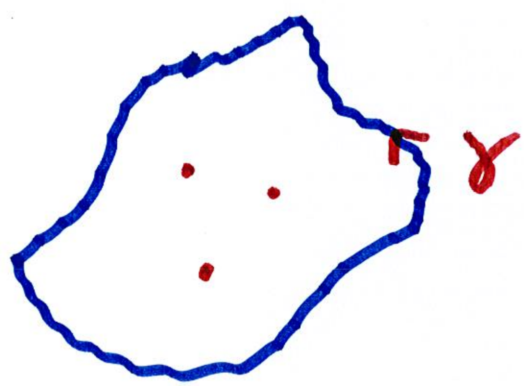
$$= \frac{2\pi i}{n} \cdot \left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}} \right)$$

$$= \frac{\pi}{n} \cdot \left(\frac{e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}}}{2i} \right)$$

$$= \frac{\frac{\pi}{n}}{\sin \frac{\pi}{n}}$$

Recall that if $f(z)$ has poles at $z_1, \ldots, z_N$ in the interior of a simple closed curve, then

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} \mathrm{Res}_{z_k} f(z).$$



If $f(z) = \sum_{k=n} c_k (z-z_0)^k \qquad c_n \neq 0$

has a zero at z_0 (so $n \geq 1$)

or has a pole at z_0, (so $n \leq -1$)

then

$$\frac{f'(z)}{f(z)} = \frac{c_n n (z-z_0)^{n-1} + \cdots}{c_n (z-z_0)^n + \cdots}$$

$$= \frac{n}{z-z_0} + \sum_{k=0}^{\infty} d_k (z-z_0)^k.$$

Thus, $\frac{f'(z)}{f(z)}$ has a

simple pole at z_k with

residue $= n_k$

Thus the Residue Formula

implies

$$\int_\gamma \frac{F'(z)\,dz}{F(z)}$$

$$= 2\pi i \left(\text{Number of Zeroes} - \text{Number of Poles} \right)$$

Rouche's Thm.

Suppose that f and g are holomorphic in a neighborhood of γ and its interior, and that $|f(z)| > |g(z)|$ for all z in γ.

Then f and $f+g$ have the same number of zeros inside γ.

Pf. For $t \in [0,1]$, define

$$f_t(z) = f(z) + t\,g(z)$$

so $f_0(z) = f(z)$ and

$$f_1(z) = f(z) + g(z)$$

Let n_t be the number of

zeros of f_t, counting multiplicities

Note that $f_t(z) \neq 0$ on γ,

since $|f| > |g|$ on γ.

Note also that

$$n_t = \frac{1}{2\pi i} \int_\gamma \frac{f_t'(z)}{f_t(z)}\, dz.$$

Since f_t is jointly continuous in z (in γ) and t.

which implies that n_t is a continuous function in t. Since n_t is integer-valued, it follows that $n_t = n_0$ for all $t \in [a, b]$. Hence

$n_1 = n_0 \Rightarrow f+g$ and f have the same number of zeros.

Ex. Let $p(z) = z^8 - 5z^3 + z - 2$

How many roots of p are in the unit circle C_1?

Set $f(z) = -5z^5$

and $g(z) = z^8 + z - 2.$

Clearly $|g| < |f|$ on C_1.

Since number of roots of f inside C_1 is $= 3$, it

Hence f and P have the same

number of zeroz inside C

(namely 3)

Ex. Find the number of zeros

of the polynomial are in the

annulus $1 < |z| < 2$, if

$P(z) = 2z^5 - 6z^2 + z + 1 = 0.$

Set $f(z) = 6z^2$, then

set $g(z) = 2z^5 + z + 1 = 0$.

Clearly f has 2 roots in C_1

~~and~~ $\cancel{|g(z) + f(z)|}$

$|g(z)| \le 4$ on C_1 and

$|f(z)| = 2$ on C_1.

$\therefore$ $f+g$ has same number in

C_1, i.e. 2 roots.

Now set $f(z) = 2z^5$

and $g(z) = -6z^2 + z + 1$

Note that $|g(z)| \leq 24 + 2 + 1$

$= 27$ on C_2

and $|f(z)| = 64$ on C_2

$\therefore$ All 5 roots of $f + g = P$

satisfy $|z_k| < 2$.

Thm. (Maximum Modulus Principle)

Suppose $f \in A(\Omega)$, and that

$$M = \sup_{z \in \Omega} \{ |f(z)| ; z \in \Omega \}.$$

and that Ω is connected

Then $|f(z)| < M$

for all $z \in \Omega$.

Clearly $|f(z)| \leq M$.

If $z_1 \in \Omega$ and $f \not\equiv C$, then

If the theorem is not true,

then $\exists$ z_1 so $|f(z_1)| = M$.

But the open mapping implies

that $f(\Omega)$ must include an

open set of $f(z_1)$, which is

impossible.

Schwartz Lemma:

Let f be holomorphic on the unit disc, $|z|<1$. Assume that $f(0)=0$ and that

$$|f(z)|<1 \quad \text{for } |z|<1.$$

Then

(1) $|f(z)| \le |z|$ for $|z|<1$

(2) If $\exists\ z_0 \neq 0$ with $|f(z_0)| = |z_0|$, then

$$f(z) = \alpha z, \quad \text{all } z \in D,$$

where $|\alpha|=1$.

f has the expansion

$$f(z) = a_1 z + \cdots$$

Then $\frac{f(z)}{z}$ is holomorphic in D_1

and

(1) $$\left| \frac{f(z)}{z} \right| < \frac{1}{r} \quad \text{for } |z| = r < 1$$

The Max. Mod. Thm implies

this inequality holds for all

$z \in \overline{D}_r$. Let $r \to 1$, then (1) holds.

If $\exists\, z_0$ so

$$|f(z_0)| = |z_0|,$$

then $g(z) = \dfrac{f(z)}{z}$ satisfies

$|g(z_0)| = 1$, and $|g(z)| \leq 1$

for all $z \in D$

$\therefore$ $g(z) = \alpha$ for some α

with $|\alpha| = 1$

$\therefore$ $f(z) = \alpha z$