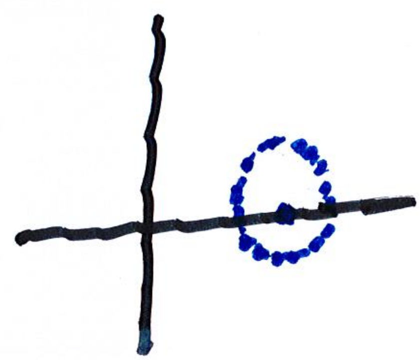
More consequences of Power Series Expansion.

Show $f(z)$ is an open mapping.

Recall $\log w$ is defined

if $w \in D_a(1)$
$(a<1)$



If $f(z) = 1 + E(z)$

where $E(z) = \sum_{n=1}^{\infty} a_n z^n$

If $|z|$ is sufficiently small,

then $|E(z)| < a$.

Now suppose that

$$f(z) = z^m + \sum_{k=m+1}^{\infty} b_k z^k .$$

Hence $$\frac{f(z)}{z^m} = 1 + \underbrace{\sum_{k=1}^{\infty} b_{k+m} z^k}_{}$$

If $|z| < r$ for some small r,

then $|E(z)| < a$

Hence $\log w$ is defined

in $D_0(1)$.

Now define $w^{1/m}$ by

$$w^{\frac{1}{m}} = \exp\left(\frac{1}{m}\log w\right)$$

Hence,

$$f_m(z) = \left(\frac{f(z)}{z^m}\right)^{\frac{1}{m}} = \exp\left(\frac{1}{m}\log\right.$$

$$= \exp\left(\frac{1}{m}\log(1+E(z))\right).$$

$$\Rightarrow (f_m(z))^m = \exp(\log(1+E(z)))$$

$$= \frac{f(z)}{z^m}.$$

Hense, $f_m(z)$ is defined and holomorphic in $D_a(0,1)$

Now set

$$g_m(z) = a_m^{1/m}\, z \cdot f_m(z).$$

Then $g_m(z)^m = a_m z^m \cdot \dfrac{f(z)}{z^m}$

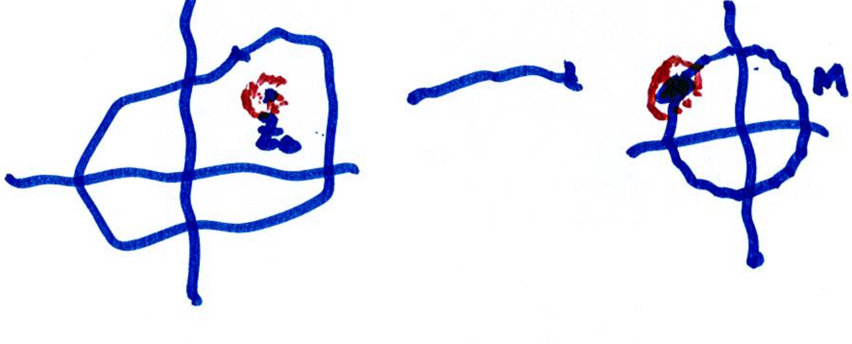
$$= f(z).$$

Corollary: Suppose $f(z)$ is holomorphic on Ω, and that $\exists z_0 \in \Omega$ and

$$|f(z_0)| = \sup\{|f(z)|;\ z \in \Omega\} = M$$

Then $f(z) \equiv C$.

$\hookrightarrow |f(z)| \leq M \quad |f(z_0)| = M$



Laurent Series.

This is a series such as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

which converges on $A = \{z : r < |z| < R\}$

we write

$$f^+(z) = \sum_{n \geq 0} a_n z^n \qquad f^-(z) = \sum_{n<0} a_n z^n$$

If the terms in f^+ and f^-

converge absolutely on A,

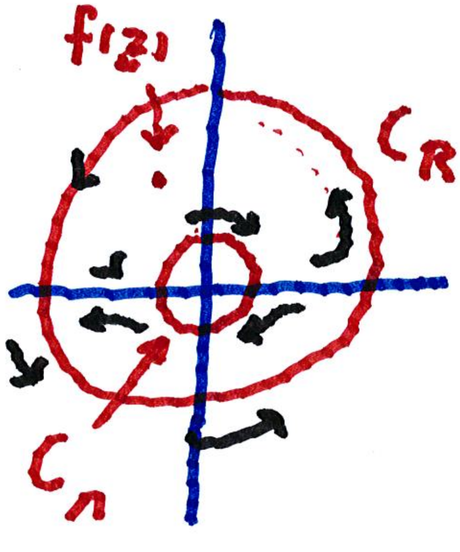
then we shall say the

Laurent series converges.

Thm. Let A = above annulus, and assume $r < s < S < R$. If f is holomorphic on $\bar{A}$, then the series converges absolutely and uniformly when $\underset{r}{\cancel{s}} \le |z| \le \underset{R}{\cancel{S}}$. The coefficients a_n satisfy

$$a_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \qquad \text{if } n \geq 0$$

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \qquad \text{if } n < 0$$



It follows that if $r \le |z| \le R$,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

For Integral 2,

$$-(\zeta - z) = z\left(1 - \frac{\zeta}{z}\right)$$

$$\therefore \frac{-1}{\zeta - z} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n$$

Since $f(z)$ is actually

holomorphic on a larger annulus

$A' = \{ z ; r-\varepsilon \leq |z| \leq R+\varepsilon \}$.

we have $\left| \frac{\zeta}{z} \right| \leq \frac{r-\varepsilon}{r} < 1$, so

the geometric series.

$$\frac{1}{z} \cdot \frac{1}{1-\frac{\zeta}{z}} = \frac{1}{z}\left(1 + \frac{\zeta}{z} + \left(\frac{\zeta}{z}\right)^2 + \cdots\right)$$

$$= \sum_{n<0} \frac{z^n}{\zeta^{n+1}}$$ converges absolutely and uniformly

Hence, we can integrate

term by terms and we get:

$$-\frac{1}{2\pi i}\sum_{n<0}\int_{C_\rho}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\cdot z^n$$

Integral 1 is handled in

the usual way:

$$\frac{1}{\zeta-z}=\frac{1}{\zeta}\left(\frac{1}{1-\frac{z}{\zeta}}\right)=\frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n.$$

Ex. Find the Laurent series

for $f(z) = \dfrac{1}{z(z-1)}$ for $0 < |z| < 1$.

First, we write $f(z)$ in partial

fractions:

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$\rightarrow 1 = A(z-1) + Bz$$

$\therefore -A = 1$, $A + B = 0 \rightarrow \underline{B = 1}$

$\underline{A = -1}$

$$\therefore f(z) = -\frac{1}{z} + \frac{1}{z-1}$$

Also, $\dfrac{1}{z-1} = -\dfrac{1}{1-z} = -(1+z+z^2+\dots)$

$$\therefore f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 \dots,$$

which converges if $0 < |z| < 1$.

Now suppose $|z| > 1$.

We can write

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right)$$

Hence

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

which converges if $|z| > 1$.

Ex. Find the Laurent Series

of $e^{(1/z)}$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad \text{converges for any } u$$

$$\therefore e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \cdot \frac{1}{n!}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \cdots$$

This function has an isolated singularity that is essential

Ex. An analytic isomorphism f is a holomorphic function $f: U \to V$ such that f is injective and surjective. By the inverse function, there is an analytic isomorphism $g: V \to U$. If $f: U \to U$ is an isomorphism, we say f is an analytic automorphism.

We want to describe all

analytic automorphisms.

Recall that if $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$,

then φ_a is an analytic automorphism.

Recall that if $|z|=1$, then

$|\varphi_a(z)| = 1$, where $|a|<1$.

By the Maximum Modulus Thm.

if $|z| < 1$, then $|\varphi_a(z)| < 1$

(since $\varphi_a \neq$ constant.)

Thus $\varphi_a : D \to D$.

Now by direct calculation.

$\varphi_{-a} \circ \varphi_a = Id_D$ and

$\varphi_a \circ \varphi_{-a} = Id_D$. This shows

φ_a is surjective, for if $|w| < 1$, then $\underbrace{\varphi_a(\varphi_{-a}(w))}$ $= w$.

It also follows easily that

φ_a is injective

Now suppose $f : D \to D$ is

an analytic automorphism

and that $f(a) = 0$. Note

that $\varphi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z}$ satisfies

$\varphi_{-a}(0) = a$. Set $g(z) = f(\varphi_{-a}(z))$

Then $g(0)=0$, so the Schwarz

Lemma implies that

$g(z) = e^{i\theta} z$, or that

$f(\varphi_{-a}(\varphi_a(z)))$

$f \circ \varphi_{-a}(z) = e^{i\theta} z$.

Replacing z by $\varphi_a(z)$, we get

$$f(z) = e^{i\theta} \varphi_a(z), \qquad |z| < 1.$$

Recall that a function $f(z)$ [illegible] with an isolated singularity at z_0 has a pole if it can be written as

$$f(z) = \frac{a_n}{(z-z_0)^n} + \dots + \frac{a_1}{(z-z_0)} + E(z)$$

where E is holomorphic near z_0

The Laurent expansion of f in the neighborhood of a singularity z_0 has only a finite number of terms

$$f(z) = \frac{a_{-m}}{(z-z_0)^{-m}} + \dots + \frac{a_{-1}}{z-z_0} + \dots$$

and $a_{-m} \neq 0$.

Thm. The only analytic automorphisms of $\mathbb{C}$ are the functions of the form

$f(z) = az + b$, where a and b are constants, $a \neq 0$

Pf. f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

If we let $w = \frac{1}{z}$, then

the function

$$h(z) := f(1/z) = \sum_{n=0}^{\infty} a_n (1/z)^n$$

for $z \neq 0$. The function h has

an isolated singularity at $z = 0$.

If the singularity is

essential, then Casorati-

Weierstrass implies that

its values (of h) come arbitrarily close to any point, in particular, close to 0.

Let D = unit disc, and let g be the inverse fcn. of f. Since $g(D)$ is open, and $g(\bar{D})$, there is $R>0$ so that $g(\bar{D})$ is contained in $\overline{D}_R$. Let U be the complement of $\overline{D}_R$. Then U is open and $f(1/z)$ for $z \in U$ does not lie in D.