We proved the Cauchy Integral Formula:

If f is holomorphic in an open set Ω and if the closed disk $\overline{D}_R(z_0) \subset \Omega$, then for any z in the interior of D_R,

$$(1) \qquad f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

In fact, for every $n = 0, 1, \ldots$,

f has derivatives of all orders

and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}},$$

$C = \partial D_R$

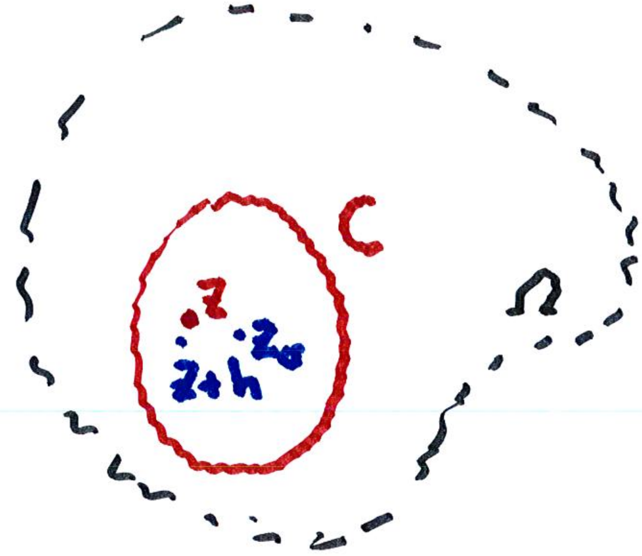
Note the formula comes from differentiating $\frac{1}{(\zeta - z)}$. Is this OK?

We prove this by induction.

We already know the case $n = 0$.

So assume

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n}}$$



If h is small, we can form the difference quotient

$$(2)\quad \frac{f^{(n-1)}(z+h) - f^{(n)}(z)}{h}$$

$$= \frac{(n-1)!}{2\pi i}\int_C \frac{f(\zeta)}{h}\left[\frac{1}{(\zeta-z-h)^n} - \frac{1}{(\zeta-z)^n}\right] d\zeta$$

Recall the formula

$$A^n - B^n = (A-B)\left[A^{n-1} + A^{n-2}B + \dots + B^{n-1} \right]$$

With $A = \dfrac{1}{(\xi - z - h)}$ and $B = \dfrac{1}{\xi - z}$.

the terms in brackets become

$$(3) \quad \frac{h}{(\xi - z - h)(\xi - z)} \left[A^{n-1} + A^{n-2}B + \dots + B^{n-1} \right]$$

Note that z and $z+h$ stay at a positive distance away from the boundary circle.

Note also that the factors of h cancel out when (3) is inserted into (2). Taking the limit as $h \to 0$, we see the quotient approaches

$$\frac{(n-1)!}{2\pi i} \int_C f(\zeta) \left[\frac{1}{(\zeta - z)^2} \cdot \frac{n}{(\zeta - z)^{n-1}} \right] d\zeta$$

$$= \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta .$$

Corollary: (Cauchy Inequalities)

Under the hypothes of the previous theorem

$$|f^{(n)}(z_0)| \le \frac{n! \, \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$

We apply the Cauchy Inequalities for $f^{(n)}(z)$ with $z = z_0$

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \right|$$

$$= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} Rie^{i\theta} \right|$$

$$\leq \frac{n!}{2\pi} \frac{\|f\|_C}{R^n} 2\pi$$

$$= \frac{n! \, \|f\|_C}{R^n}$$

Now we show that f can be written as a power series in $D_R(z_0)$.

Thm. Suppose that f is holomorphic in Ω and that $\overline{D}_R(z_0) \subset \Omega$. Then f has a power series expansion at z_0. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, for all $z \in D_R(z_0)$. Also, the coefficients are:

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad \text{for all } n \geq 0$$

Pf. By the Cauchy Integral Formula.

if $z \in D_R(z_0)$ and if $\zeta \in C_R$,

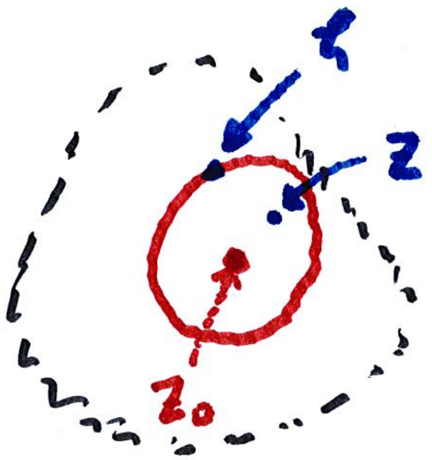
$$(4) \quad \frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}$$

Since $\zeta \in C_R$ and $z \in D_R(z_0)$

there is an r so $0 < r < 1$

such that

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r .$$



Hence, we can

write

$$(5) \qquad \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0} \right)} = \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{\zeta - \zeta} \right)^n$$

where the sum converges

uniformly

for all $z \in D_{rR}(z_0)$

By combing (1), (4) and (5)

we conclude that

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right) \cdot (z - z_0)^n$$

which proves the theorem.

Some Applications

We say a function $f(z)$ is **entire** if it is holomorphic in all of the complex plane.

Thm.
Suppose that f is a bounded entire function. Then f is a constant.

If f is bounded, then there is $M > 0$ so that

$$|f(z)| \leq M \quad \text{for all } z \in \mathbb{C}$$

For any $R > 0$, Cauchy's Estimates imply that for any z_0

$$|f'(z_0)| \leq \frac{M}{R},$$

Letting $R \to \infty$, it follows that

$F'(z_0) = 0$ for all z_0,

which implies that f is a constant.

Corollary. Every non-constant polynomial $P(z)$ with complex coeffients has a root.

Proof: If $p(z) = \sum_{k=0}^{n} a_k z^k$.

and if we WLOG assume that $a_n \neq 0$.

Then

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$$

then for sufficiently large z,

say $|z| \geq R$, we can assume

that $\frac{P(z)}{z^n} \geq \frac{|a_n|}{2}$. or

(6) $$|P(z)| \geq \frac{|a_n| R^n}{2}, \text{ if } |z| \geq R.$$

On the other hand, since

$\overline{D}_R(0)$ is compact and $P(z) \neq 0$,

there is $m > 0$ so that

$|p(z)| \geq m$, for all $z \in \overline{D}_R(0)$.

or equivalently that

$\frac{1}{p(z)} \leq \frac{1}{m}$ for all $z \in D_R$

Thus, we have shown that

$\frac{1}{p(z)}$ is a bounded, which

by Liouville's implies that

p is a constant, which is

a contradiction.

Corollary. Every polynomial of degree $n \geq 1$ has n (possibly repeated) roots $w_1, \ldots, w_n$, so that

$$P(z) = a_n (z - w_1) \ldots (z - w_n)$$

Thm. Suppose f is holomorphic on an open set Ω, and that there is a sequence of distinct points z_n, $n=1,\dots,$ in Ω

such that $\lim_{n\to\infty} z_n = z_0$,

where z_0 is also in Ω.

Then $f(z)=0$ on all of Ω

Pf. By theorem on power series,

there is a convergent series

$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. By continuity

$f(z_0)$, i.e., a_0 is $=0$.

Let $m \geq 1$ be the smallest integer such that $a_m \neq 0$.

Then we can write

$$f(z) = (z - z_0)^m g(z),$$

where $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$,

and $b_n = a_{n+m}$. Since $g(z_0) = a_m$,

it follows that there is $r > 0$ so that $|g(z)| > 0$, when $|z| < r$

The points $\{z_n\}$ are distinct, so there is $N>0$ so that $z_k \neq 0$ for all $k \geq N$. Since $\lim_{k\to\infty} f(z_k) = 0$, $f(z_k) \neq 0$ for all large k.

This contradiction shows that $a_n = 0$ for all n. It follows that $f(z) = 0$ for all z in a neighborhood

Assuming that Ω is connected.

let $U = \{ z \in \Omega;\ f$ vanishes in an open set about $z \}$

Clearly, U is open.

But we showed that if $\{z_k\}_{k=1}^{\infty} \subset U$ which converges to a point $\tilde{z} \in U$, then $\tilde{z} \in U$

Hence U is closed.
Since U is nonempty, open and closed, it must be that $U = \Omega$.

Hence $f \equiv 0$ in Ω.