Lesson 9.

Here is a converse of Gourat's Thm. for triangles.

Thm. Suppose f is continuous on an open disc and that

$$\int_T f(z)\,dz = 0$$

for every triangle $T \subset D$. Then f is holomorphic.

Fix $z_0 \in D$. Then for every z in D, set $F(z) = \int_{\gamma_z} f(z)\,dz$,

where γ_z is the straight path from z_0 to z. Then if h is small,

$$F(z+h) - F(z) = \int_{\gamma_{z,z+h}} f(z)\,dz$$

where $\gamma_{z,z+h}$ is the straight path from z to $z+h$. As before, since f is continuous at z,

$$\int_{\gamma_{z,z+h}} f(w)\,dw = hf(z) + |h|\,\psi_1(h).$$

where $|\psi_1(h)| \to 0$ as $h \to 0$

$$\therefore \quad \frac{F(z+h) - F(z)}{h} = f(z) + |\psi_1(h)|$$

Taking the limit as $h \to 0$,

$F'(z) = f(z)$, so F is holomorphic in D.

and F is twice differentiable

Thus the Regularity Thm implies

that F is twice differentiable,

i.e. $$\lim_{h\to 0} \frac{F^{(1)}(z+h) - F^{(1)}(z)}{h}$$

$$= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$ exists.

Thus f is holomorphic in D.

Sequences of Holomorphic Functions

Thm. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function on each compact subset of Ω, then f is holomorphic in Ω.

Pf. Let D be any disc with $\bar{D} \subset \Omega$, and let T be any

any triangle in D. Since f_n is holomorphic in D, $\int_T f_n(z)\,dz = 0$.

By assumption, $f_n \to f$ uniformly on T, so $\int_T f(z)\,dz = 0$

Morera's Thm implies that f is holomorphic in D.

Since D is any disc with $\bar{D} \subset \Omega$, the theorem follows.

More generally, we can show that $f_n^{(k)} \to f^{(k)}$ uniformly on any compact set. We note that if

$$\tilde{K}_\delta = \Big\{ z ; \ d(z, b\Omega) \geq 2\delta \text{ and } |z| \leq \frac{1}{\delta} \Big\}$$

then any compact set K is contained in $\tilde{K}_\delta$ for some δ.

Also, if

$$K_\delta = \left\{ \zeta ; \begin{array}{l} d(\zeta, b\Omega) \geq \delta \\ \text{and } |\zeta| \geq \frac{1}{\delta} + 1 \end{array} \right\},$$

then any compact set $K_{\cancel{\delta}}$ is $\subset K_\delta$ for some δ.

Suppose that $z \in K'_\delta$. The Cauchy Integral Formula for k-th derivatives states that

$$f_n^{(k)}(z) - f^{(k)}(z)$$

$$= \frac{k!}{2\pi i} \int_{C_\delta(z)} \frac{\{f_n(\zeta) - f(\zeta)\}\, d\zeta}{(\zeta - z)^{k+1}} .$$

Hence $\left| \left(f_n^{(k)} - f^{(k)} \right)(z) \right|$

$$\leq \frac{1}{2\pi} \int_{C_\delta(z)} \frac{k!\, \left| \left| f_n(\zeta) - f(\zeta) \right| \right| d\zeta|}{|\zeta - z|^{k+1}}$$

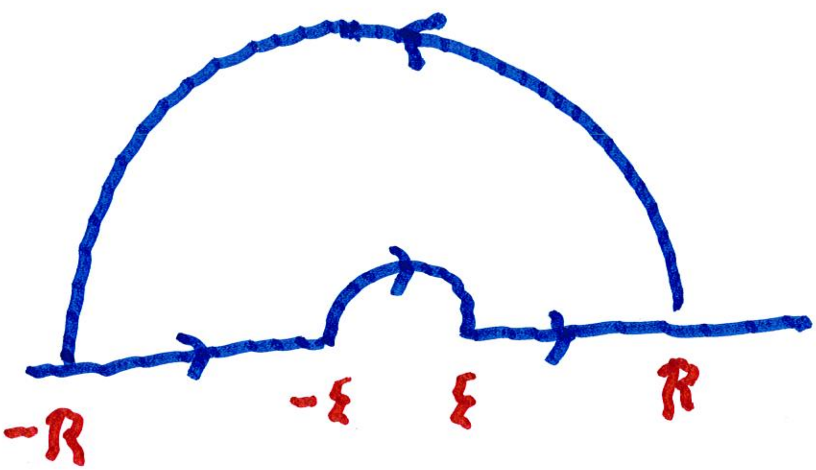
$$\leq \frac{1}{2\pi} \int \frac{k! \sup_{\zeta \in K_\delta} |f_n(\zeta) - f(\zeta)|}{\delta^{k+1}} 2\pi\delta$$

$$= \frac{k! \sup_{\zeta \in K_\delta} |f_n(\zeta) - f(\zeta)|}{\delta^k}.$$

We can use holorphic line integrals to compute definite integrals

Compute $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx$.

We use the path



$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}$$

$$+ \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0$$

First let $R \to \infty$.

If $z = x + iy$, where $y \geq 0$,

then $|e^{i(x+yi)}| = |e^{ix} \cdot e^{-y}|$

Hence $\displaystyle\int \frac{|1 - e^{iz}|}{R^2} |dz|$

$$\leq \int_0^{\pi} \frac{2}{R^2} |iRe^{i\theta}| d\theta \to 0$$

as $R \to \infty$