Inhomogeneous Cauchy-Riemann Equation.

Basics:

Since $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$

or, if u is a C^1 fcn,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\rightarrow du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z},$$

where $\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right)$

and $\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right)$

A fcn. $U + iV$ is holomorphic

if $\bar{\partial}(U+iV) = 0$.

or $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(U+iV)$

$= \frac{1}{2}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) = 0$

(by the Cauchy - Riemann Eq'ns.)

$du = \cancel{\partial u}$

$= \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}$

Let ω be an open set in $\mathbb{C}$.

By Stokes' Formula

$$\int_{\partial\omega} f\,dx + g\,dy = \iint_{\omega} d\{f\,dx + g\,dy\}$$

$$= \iint_{\omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

or $$\int_{\partial\omega} v\,dz = \iint_{\omega} dv \wedge dz = \iint_{\omega} \bar{\partial} v \wedge dz$$

Note that

$$\bar{\partial} v \wedge dz = \frac{\partial v}{\partial \bar{z}} \, d\bar{z} \wedge dz$$

$$= 2i \, \frac{\partial v}{\partial \bar{z}} \, dx \wedge dy$$

$\therefore$

$$(1) \quad \int_{\partial \omega} v \, dz = \iint_{\omega} \frac{\partial v}{\partial \bar{z}} \, d\bar{z} \wedge dz.$$

Thm. If $v \in C^1(\bar{\omega})$, then

$$u(\zeta) = \frac{1}{2\pi i}\left\{ \int_{\partial \omega} \frac{u(z)}{z-\zeta}\, dz + \iint_{\omega} \frac{\frac{\partial u}{\partial \bar{z}}}{z-\zeta}\, dz \wedge d\bar{z} \right\}$$

In homogeous Cauchy-Riemann Formula.

Put $\omega_\varepsilon = \{ z \in \omega ;\ |z-\zeta| \geq \varepsilon \}$

Now apply (1) to $\frac{u(z)}{z-\zeta}$. We get

$$-\int_0^{2\pi} u(\zeta+\varepsilon e^{i\theta})\, i\,d\theta + \int_{\partial w} \frac{u(z)}{z-\zeta}\,dz$$

$=$

$$= \iint_{w_\varepsilon} \frac{\frac{\partial u}{\partial \bar z}}{z-\zeta}\, d\bar z \wedge dz$$

The term on the left is

$z = \zeta + \varepsilon e^{i\theta}$

$$-\int_0^{2\pi} \frac{u(z)\,dz}{z-\zeta}$$, which is parameterized

by $z = \zeta + \varepsilon e^{i\theta}$,

$0 \le \theta \le 2\pi$

Letting $\varepsilon \to 0$, we divide by $2\pi i$, which gives the result.

Thm. If μ is a measure with compact support in $\mathbb{C}$. then the integral $u(\zeta) = \int (z-\zeta)^{-1} \, d\mu(z)$ defines a holomorphic C^∞ fcn. outside the support of μ.

In any open set where $d\mu = (2\pi i)^{-1} \phi \, dz \wedge d\bar{z}$ for some $\phi \in C^k(\omega)$, we have $u \in C^k(\omega)$ and $\frac{\partial u}{\partial \bar{z}} = \phi$, if $k \geq 1$.

Proof: That $u \in C^\infty$ outside the support K of is obvious

Since $(z-\zeta)^{-1}$ is a C^∞ fcn.

when $z \in K$ and $\zeta \in \complement K$

and since $\dfrac{\partial (z-\zeta)^{-1}}{\partial \bar{\zeta}} = 0$ when

$\zeta \neq z$, the holomorphicity ~~quantity~~ follows

by differentiation under the

integral sign.

For the second statement,

assume first that $\omega = \mathbb{R}^2$

Changing variables, we can

write $(\zeta - z) = t$.

$$u(\zeta) = -(2\pi i) \iint \phi(\zeta - z)\, z^{-1}\, dz \wedge d\bar{z}$$

Since z^{-1} is integrable on every

compact set, we can differentiate

at most k times under the integration sign ~~at most~~ and the integrals obtained are continuous. Hence $u \in C^k$ and

$$\frac{\partial u}{\partial \bar{\zeta}} = -(2\pi i) \iint \frac{\partial \varphi(\zeta - z)}{\partial \bar{\zeta}}\, z^{-1}\, dz \wedge d\bar{z}$$

$$= 2\pi i \iint (z-\zeta)^{-1} \frac{\partial \varphi}{\partial \bar{z}}(z)\, dz \wedge d\bar{z}.$$

By applying Theorem 1 with u replaced by φ and ω equal to a disc containing the support of φ gives

$$\frac{\partial u}{\partial \bar{\zeta}} = \varphi$$

Finally, if ω is arbitrary, we can for every $z_0 \in \omega$, choose a function $\Psi \in C_0^k(W)$

which is = to 1 in a neighborhood

V of z_0. If $\mu_1 = \Psi\mu$ and

$\mu_2 = (1-\Psi)\mu$, we have

$U = U_1 + U_2$, where

$$\mu_j(\zeta) = \int (z-\zeta)^{-1}\, d\mu_j(\zeta).$$

Since μ_1 is equal to

Since

$(2\pi i)^{-1}\,\psi\varphi\, dz \wedge d\bar{z}$ and

$\psi\varphi \in C_0^k(\mathbb{R}^2)$, we have

$u \in C^k(V)$ and that

$\frac{\partial u}{\partial \bar{z}} = \varphi$ in V. The proof

is complete.