The ζ function

We define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{where}$$

$$s = \sigma + it .$$

We first show that the above series converges when $\text{Re}\, s = \sigma$ and where is > 1.

Proof: If $s = \sigma + it$ where σ and t are both real, then

$$|n^{-s}| = |e^{-s\log n}| = e^{-\sigma \log n} = n^{-\sigma}.$$

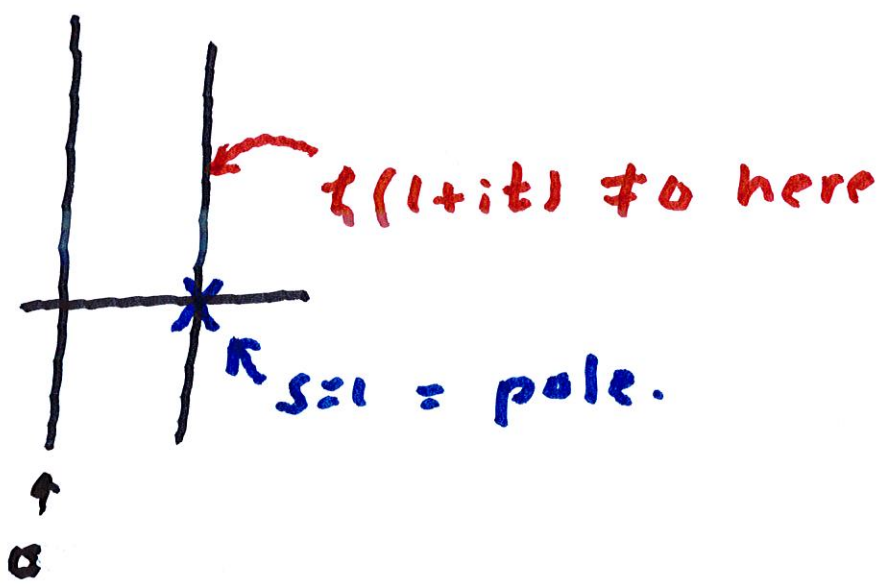
Hence, if $\sigma > 1 + \delta > 1$, the series defining ζ is uniformly bounded by

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}},$$ which converges uniformly.

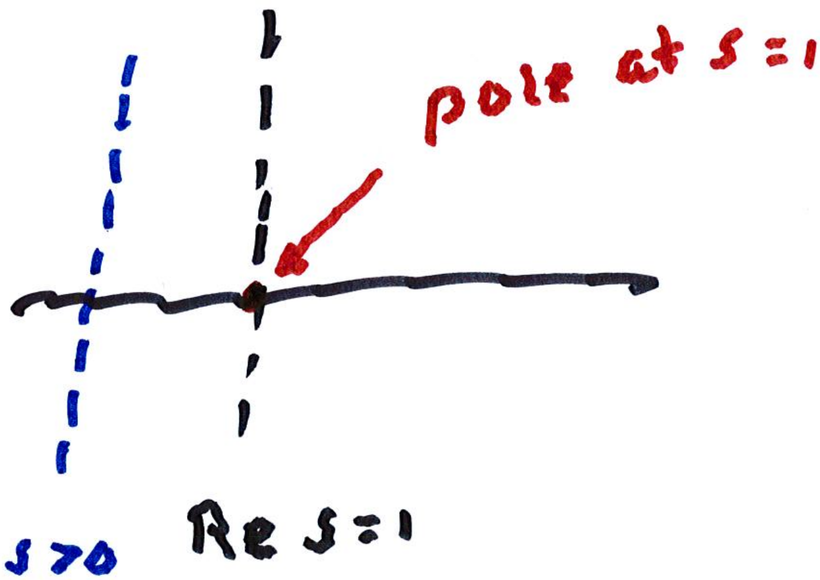
of numbers between 0 and 1

is negative.

Thus $\zeta(1+it) \neq 0$, for all t



The ζ-function extends to a meromorphic function that has a simple pole when $s=1$. We will show that is true when $\text{Re}\, s > 0$



Proposition. There is a sequence of entire fcns $\{\delta_n(s)\}_{n=1}^{\infty}$ such that

$$|\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}},$$

and such that

$$\sum_{1 \le n < N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1 \le n < N} \delta_n(s)$$

Pf. We compare $\sum_{n=1}^{N-1} n^{-s}$

with $\sum_{n=1}^{N-1} \int_n^{n+1} x^{-s}$.

Thus we set

$$\delta_n(s) = \int_n^{n+1} \left[\frac{1}{n^s} - \frac{1}{x^s} \right] dx.$$

The mean-value theorem

applied to $f(x) = x^{-s}$ yields

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| = \frac{|s|}{n^{\sigma+1}} |x-s| \leq \frac{|s|}{n^{\sigma+1}}$$

when $n \leq x \leq n+1$. Since

(1) $$\int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \int_n^{n+1} \frac{dx}{x^s},$$

the proposition is proved

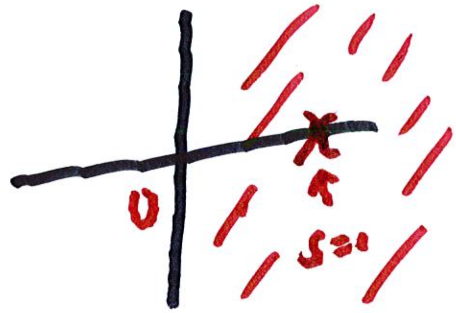
Corollary. If Re $s > 0$, then

$$\zeta(s) - \frac{1}{s-1} = H(s),$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is

holomorphic in the half-plane

$\operatorname{Re} s > 0$



Proof: First assume $\operatorname{Re}(s) > 1$.

We let $N \to \infty$ in (1). Because

$$|\delta_n(s)| \leq \frac{|s|}{n^{\sigma+1}},$$

the series $\sum_{n=1}^{\infty} \delta_n(s)$

converges [uniformly] to $H(s)$

when $Re(s) > \delta > 0$.

Again, the limit $H(s)$ must

be holomorphic in $Re(s) > 0$.

By uniqueness of extension

$\zeta(s)$ is extendable as a

meromorphic fcn. when $Re(s) > 0$.

Factorization of $\zeta(s)$.

We will show that

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \quad \text{when } \operatorname{Re}(s) > 1.$$

Recall that every integer n

can be uniquely factored as

$$n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} .$$

Let N be any integer > 0 and

let $p_1, \ldots, p_L$ be all the primes $\leq N$.

then for every $n \leq N$, we can

let $\{e_1, e_2, \ldots, e_L\}$ be an L-tuple

such that $n = p_1^{e_1} \cdots p_L^{e_L}$.

The map $n \longrightarrow L$-tuple is

1-to-1 (by uniqueness of

factorization)

Hence, we obtain

$$\sum_{n=1}^{N} \frac{1}{n^s} \le \prod_{p \le N} \left(1 + \frac{1}{p^s} + \cdots \frac{1}{p^{Ns}}\right)$$

$$\le \prod_{p \le N} \left(\frac{1}{1-p^{-s}}\right)$$

or $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ $$\le \prod_{p} \left(\frac{1}{1-p^{-s}}\right).$$

For the reversel inequality:

$$\prod_{p \le N} \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{Ns}}\right) \le \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Letting $N \to \infty$, we get

$$\prod_p \left(\frac{1}{1-p^{-s}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Thus the two quantities are equal.

Note that $\zeta(s) \neq 0$ if $Re(s) < 1$.

The critical strip is the set of points $s = \sigma + it$ such that

$0 \le \sigma \le 1$.

The Riemann Hypothesis states that if $s = \sigma + it$ is a zero in the critical strip, then $\sigma = \frac{1}{2}$

Thm. The zeta function

has no zeros on the line $Re(s)=1$.

Lemma. If $Re(s)>1$, then

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

for some $c_n \geq 0$.

Proof: Suppose $s>1$. By

the Taylor series of $\log\left(\frac{1}{1-x}\right)$

$$= \sum_{m=1}^{\infty} \frac{x^m}{m}$$

which is true for $0 \leq x < 1$.

we find

$$\log(\zeta(s)) = \log \prod_p \frac{1}{1-p^{-s}}$$

$$= \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{p,m} \frac{p^{-ms}}{m}$$

$$= \sum_{n=1}^{\infty} c_n n^{-s},$$

where $c_n = \frac{1}{m}$ if $n = p^m$ and

$c_n = 0$ otherwise.

Lemma.

If $\theta \in \mathbb{R}$, then

$$3 + 4\cos\theta + \cos 2\theta \ge 0$$

Pf. Follows from

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \ge 0$$

Corollary. If $\sigma > 1$, and t is real,

then ~~⇒~~ $\log\left|\zeta^3(\sigma)\,\zeta^4(\sigma+it)\,\zeta(\sigma+2it)\right| \ge 0$

Proof: Let $s = \sigma + it$. Then

$$\mathrm{Re}\,(n^{-s}) = \mathrm{Re}\,(e^{-(\sigma+it)\log n})$$

$$= e^{-\sigma \log n} \cos(t \log n) = n^{-\sigma} \cdot \cos(t \log n)$$

~~Hence. $\log \zeta(s) = \Sigma$~~ Hence

$$\log \left| \zeta^3(\sigma)\, \zeta^4(\sigma+it)\, \zeta(\sigma+2it) \right|$$

$$= 3 \log|\zeta(\sigma)| + 4 \log|\zeta(\sigma+it)|$$

$$+ \log|\zeta(\sigma+2it)|$$

$$= 3\,\mathrm{Re}\left[\log \zeta(\sigma)\right] + 4\,\mathrm{Re}\left[\log \zeta(\sigma+it)\right]$$

$$+ \mathrm{Re}\left[\log \zeta(\sigma+2it)\right]$$

$$= \sum_{n=1} c_n n^{-\sigma} \underbrace{(3 + 4\cos\theta_n + \cos 2\theta_n)}_{\geq 0},$$

where $\theta_n = t \log n$.

Positivity follows from Lemma

and that $c_n \geq 0$

$$\sum_{n=1}^{\infty} c_n n^{-s} = \sum_{n=1}^{\infty} c_n e^{-\ln n \cdot (\sigma + it)}$$

$$= \sum_{n=1}^{\infty} c_n n^{-\sigma} \cdot \left(\cos(t \ln n) + i \sin(t \ln n) \right)$$

Explanation.

We defined

$$\delta_n(s) = \int_n^{n+1} \left[\frac{1}{n^s} - \frac{1}{x^s} \right] dx$$

If we use the inequality

$$\left| f(a) - f(b) \right| \le |b-a| \cdot \sup_{t \in [a,b]} |f'(t)|,$$

with $f(t) = \frac{1}{t^s}$, $b = n+1$, $a = n$,

then we get

We now show that $\zeta(1+it) \neq 0$.

Suppose that $\zeta(1+it_0) = 0$.

Since ζ is holomorphic at $1+it_0$,

it vanishes to order 1 at this point,

Hence,

$$|\zeta(\sigma+it_0)|^4 \leq C(\sigma-1)^4 \text{ as } \sigma \to 1.$$

Also, $s=1$ is a simple pole, so

$$\Rightarrow |\zeta(\sigma)|^3 \leq C'|\sigma-1|^{-3}$$

as $\sigma \to 1$

Finally, since ζ is holomorphic at $\sigma + 2it_0$, the quantity

$|\zeta(\sigma + it_0)|$ is bounded.

Hence,

$$|\zeta^3(\sigma)\,\zeta^4(\sigma+it_0)\,\zeta(\sigma+2it_0)| \to 0$$

as $\sigma \to 1$.

which contradicts the above Corollary, since the logarithm