12.4 Cross Product

Suppose $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are

vectors in $\mathbb{R}^3$

We define

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \rangle$$

cross product

$$\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$$

or $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Ex. Compute $\langle 2, 1, -2 \rangle \times \langle 1, 3 -1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (-1+6)\vec{i} - (-2+2)\vec{j} + (6-1)\vec{k}$$

$$= 5\vec{i} + 5\vec{k}$$

Fact: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

and $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$\{\vec{a} \times b\} \cdot \vec{a}$

$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$

$= \{a_2 b_3 - a_3 b_2\} a_1 - \{a_1 b_3 - a_3 b_1\} a_2$

$+ \{a_1 b_2 - a_2 b_1\} a_3 = 0 .$

$\therefore$ $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

Similarly

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right)$$

Ex. Find a vector $\vec{v}$ that is

$\perp$ to $\langle 1, 2, 3 \rangle$ and $\langle 2, 2, -1 \rangle$

$\vec{a}$ $\qquad\qquad\qquad$ $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

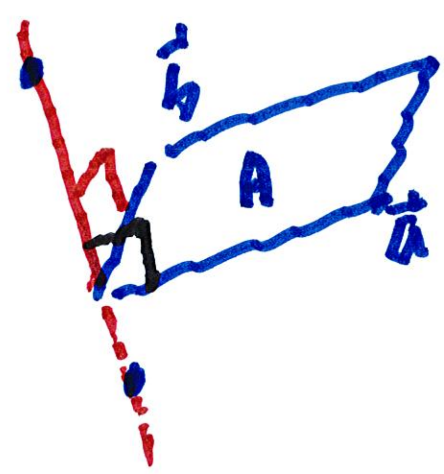
$$= \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= -8\vec{i} - (-1-6)\vec{j} + (2-4)\vec{k}$$

$$= \underline{-8\vec{i} + 7\vec{j} - 2\vec{k}} = \vec{v}$$

$\therefore$ $\vec{a} \times \vec{b}$ lies on line

that is $\perp$ to $\vec{a}$ and $\vec{b}$



Fact: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

where θ = angle between

$\vec{a}$ and $\vec{b}$.

Also, $|\vec{a} \times \vec{b}|$ = area of

parallelogram spanned

by $\vec{a}$ and $\vec{b}$

Ex. Find the area of the triangle with vertices at

$P(2,1,-1)$ $Q(1,1,2)$ and

$R(3,2,1)$.

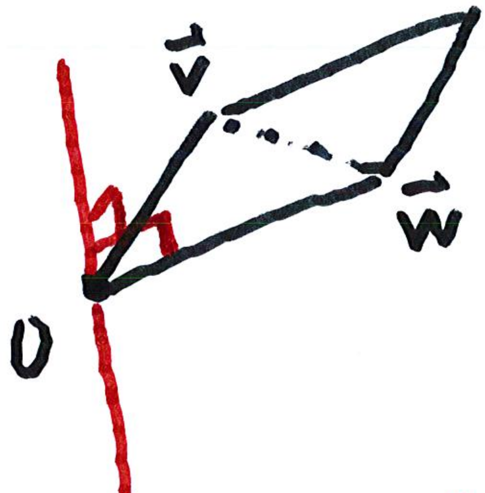
$\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$

span parallelogram,

Note: $(0,0,0)$ is now a vertex of the parallelogram.

$\vec{v} = \langle -1, 0, 3 \rangle$ and

$\vec{w} = \langle 1, 1, 2 \rangle$



$\vec{a} \times \vec{b}$ is $\perp$ to plane spanned by $\vec{v}$ and $\vec{w}$.

Area of Δ spanned by $\vec{v}$ and $\vec{w}$ $= \frac{1}{2}\left(\text{Area of Parallelogram}\right)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -3\vec{i} - (-5)\vec{j} + (-1)\vec{k}$$

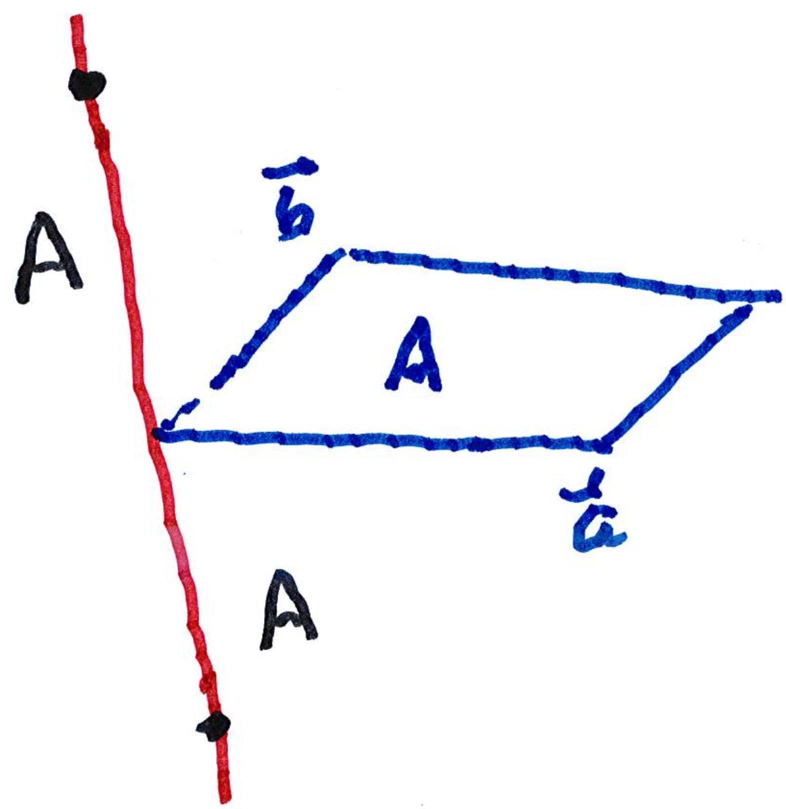
$$= -3\vec{i} + 5\vec{j} - \vec{k}$$

$$|\vec{v} \times \vec{w}| = \sqrt{9 + 25 + 1}$$

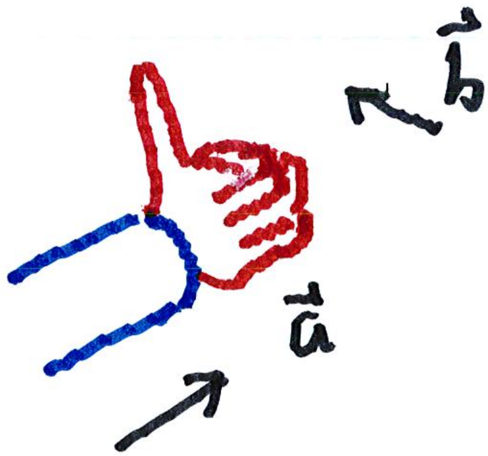
Area of $\Delta = \frac{1}{2}\sqrt{35}$

Now we know $\vec{a} \times \vec{b}$

lies at one of 2 locations



Right Hand Rule



Fingers curl from $\vec{a}$ to $\vec{b}$

$\Rightarrow \vec{a} \times \vec{b}$ points in

direction of thumb

Triple Product

Given vectors

$\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} \cdot (\vec{b} \times \vec{c})$ (the triple product)

equals volume of parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$