Math 261

Multivariate Calculus

David Catlin 744 in
Math Bldg.

Office Hours

M 9:30 – 10:30

W 10:30 – 11:30

Th 9:30 – 10:30

We use WebAssign for homework

HW from Friday and Monday Lecture is due by Tuesday by 11:00 PM.

HW from Wednesday Lecture is due by Thursday by 11:00 PM

No Makeups for HW or Quizzes

3 Lowest HW scores and

Quizzes will be dropped

Please read Ground Rules

at math.purdue.edu, then

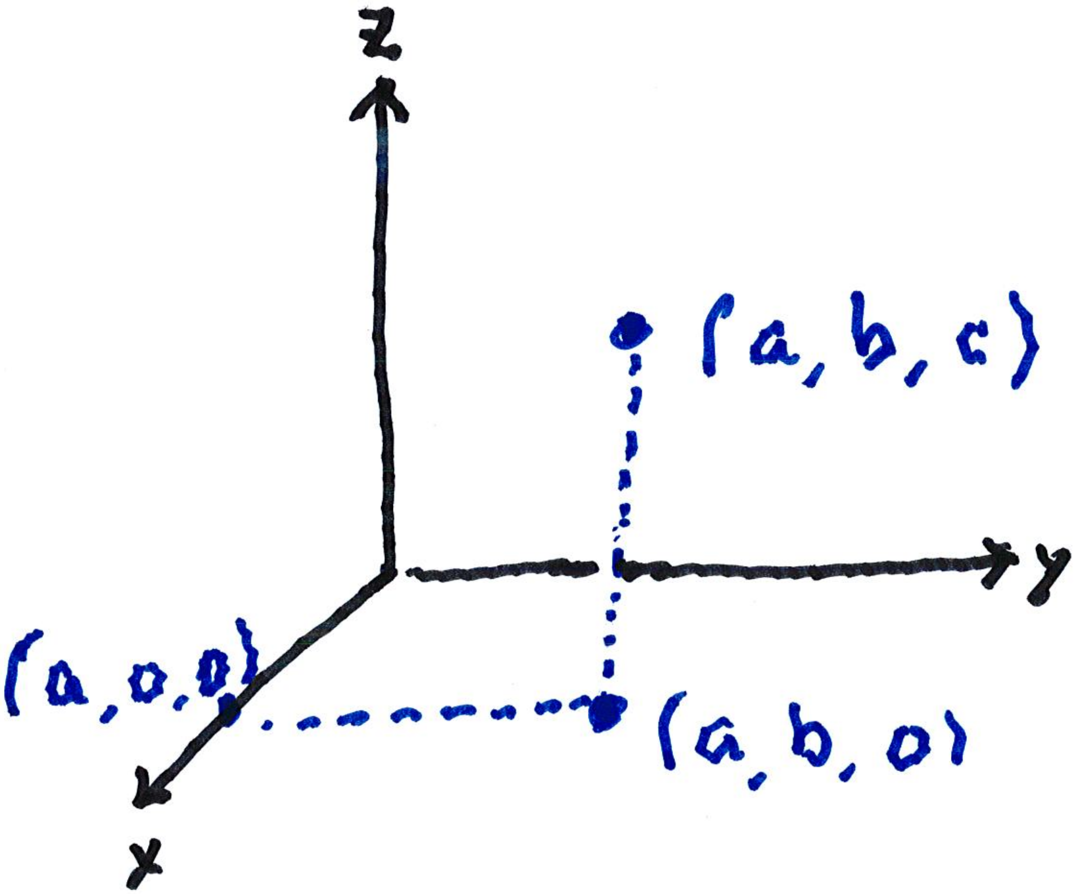
go to Course Page.

Also look at Calendar.

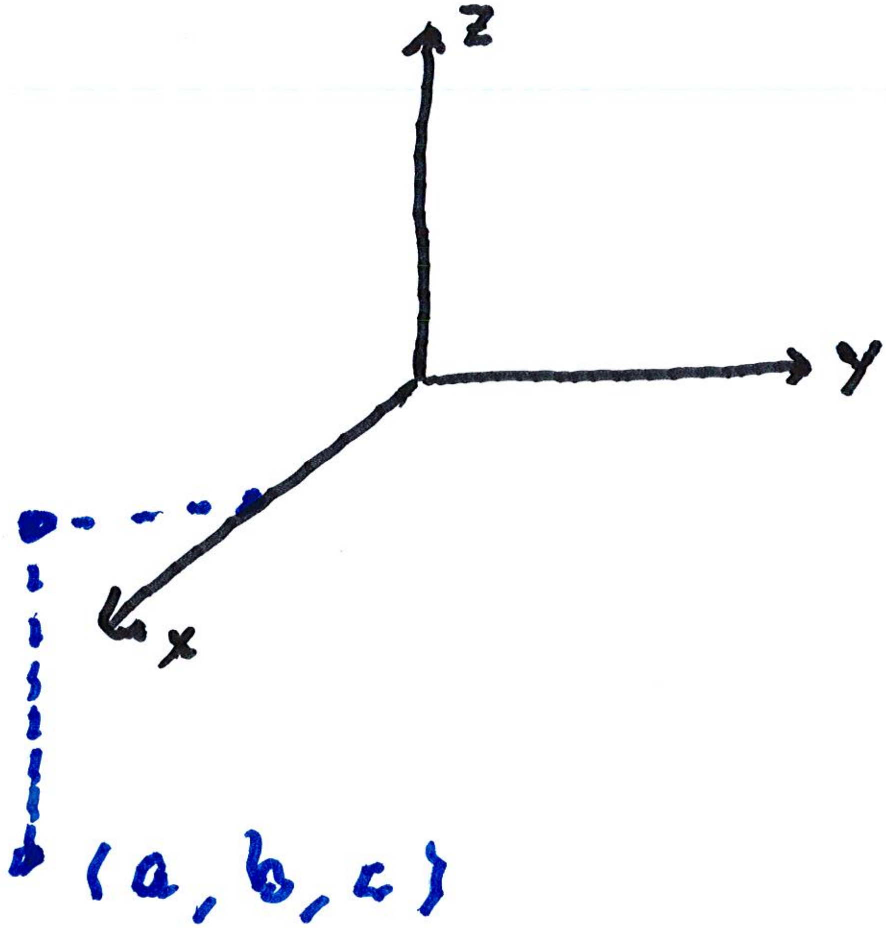
12.1 — 12.4 Review

12.1 3-dimensional coordinates

Each point in 3-dim. space can be written as $P(a,b,c)$



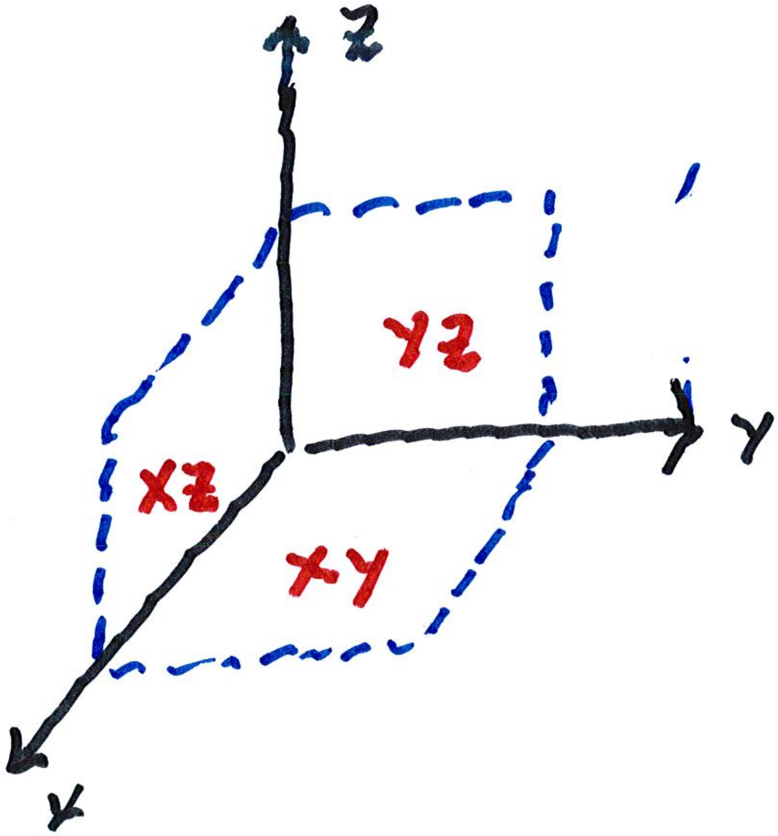
Suppose $a > 0$ and $b, c < 0$



Note that the xy-plane is $z = 0$

The xz-plane is $y = 0$

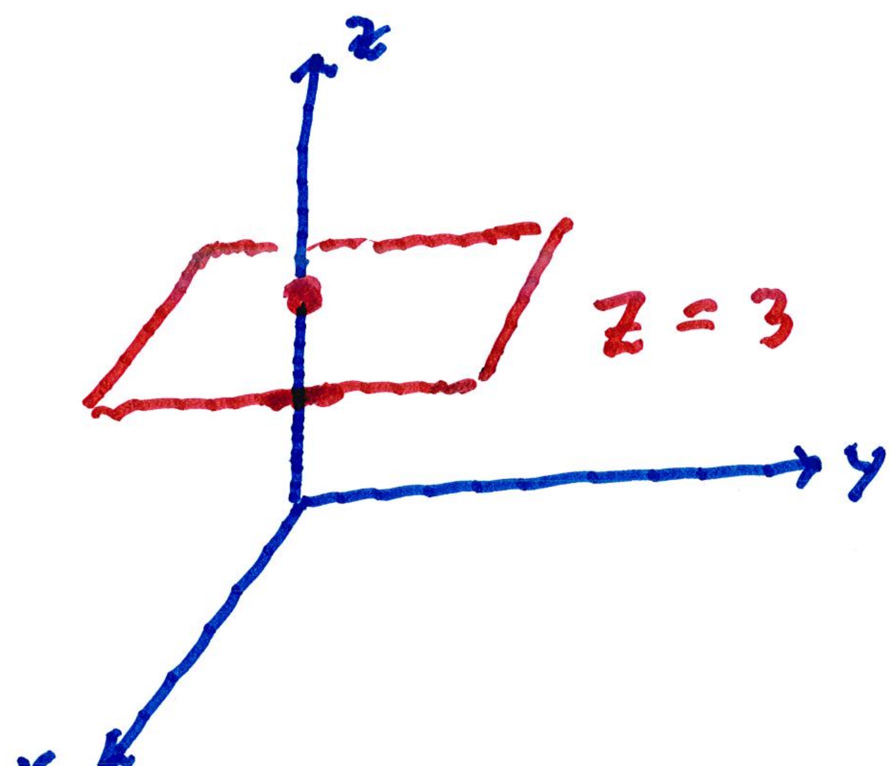
The yz-plane is $x = 0$



The equation $z = 3$

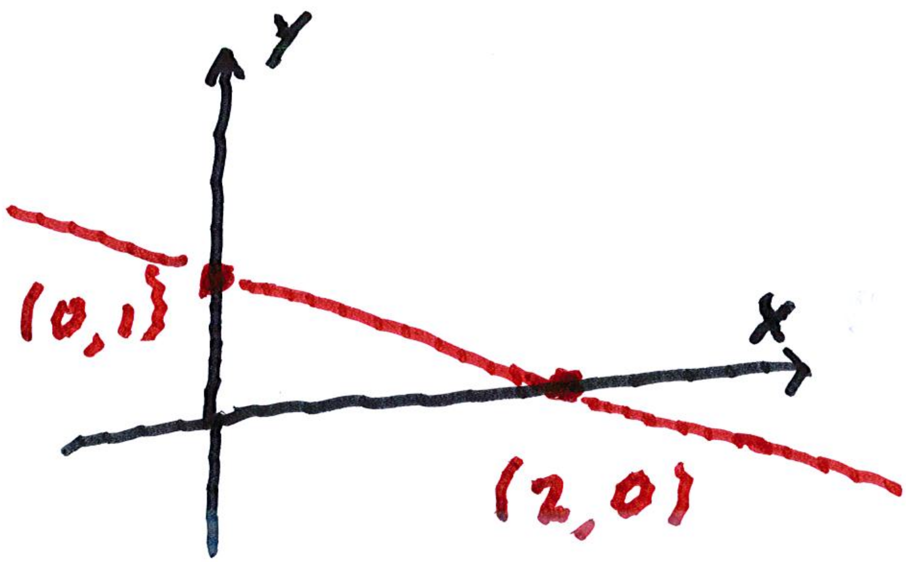
defines a flat plane

through $(0, 0, 3)$

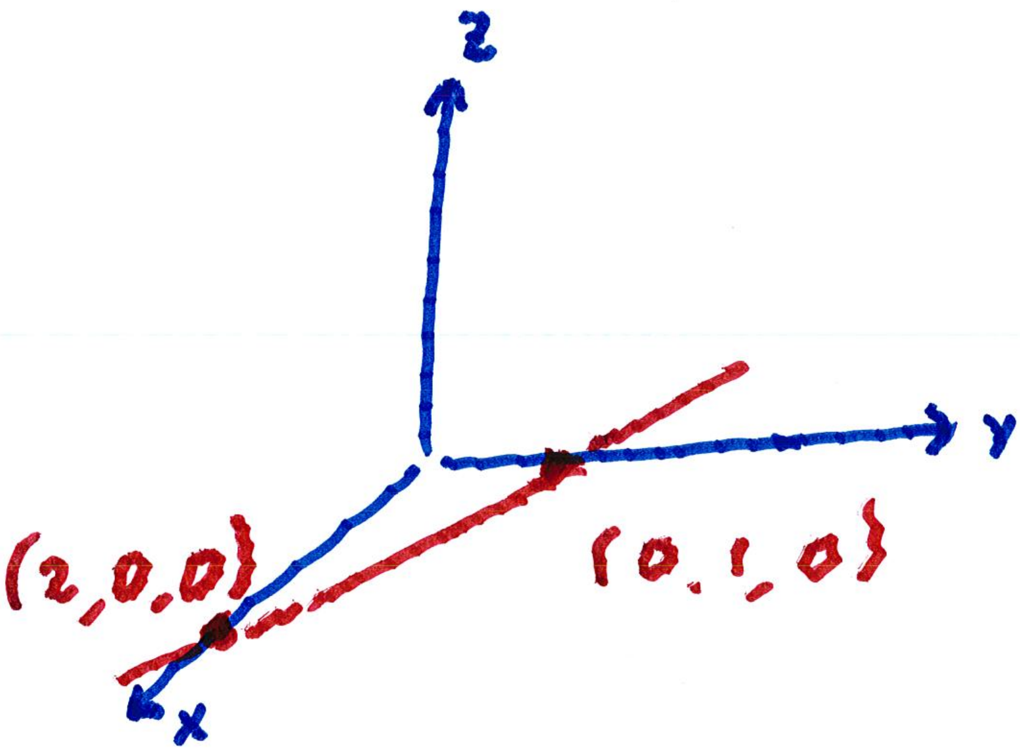


Ex. Sketch $x + 2y = 2$

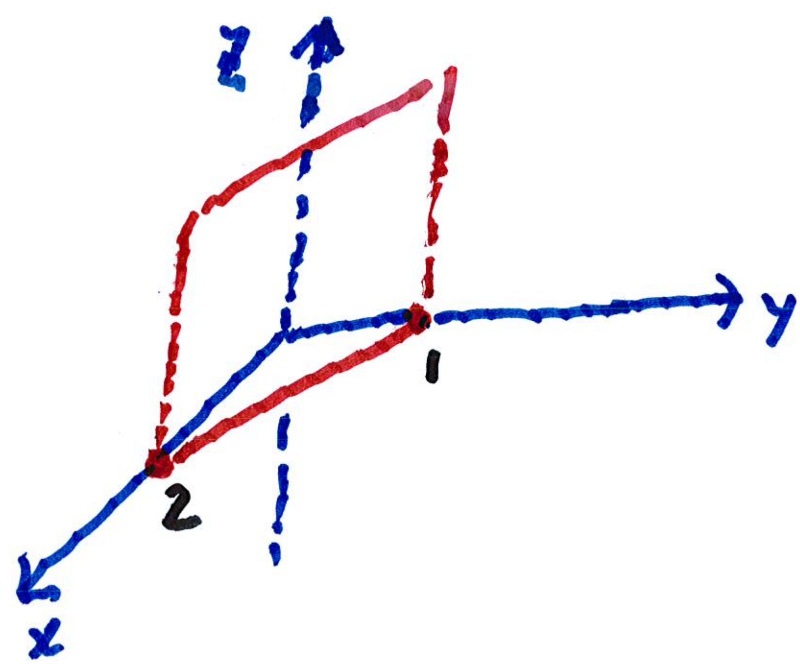
(ind. of z)



Put this line in xy-plane:

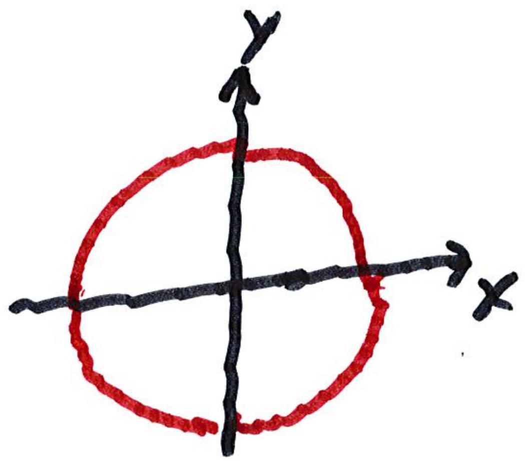


Slide the line up and down



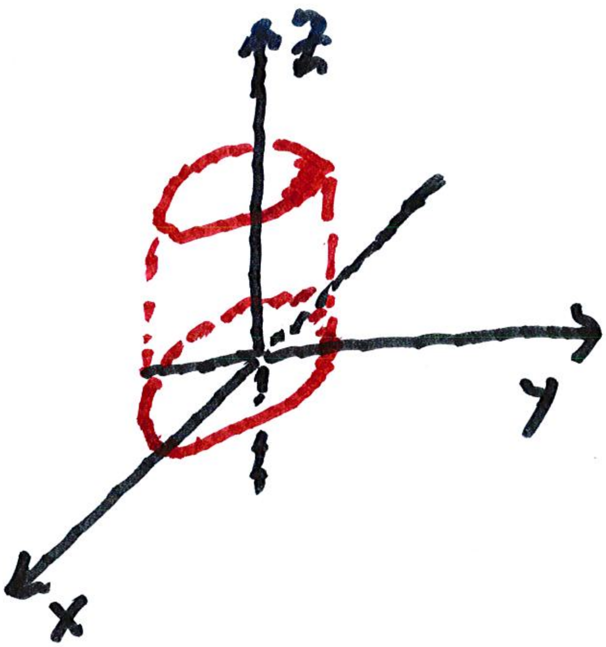
Ex. Sketch $x^2 + y^2 = 4$

Same idea:



$x^2 + y^2 = 4$

(radius = 2)



(ind. of z)

The distance between

$P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Ex. Find the radius and center of

$$x^2+y^2+z^2+2x-4y+4z = 3$$

$$(x+1)^2+(y-2)^2+(z+2)^2$$

$$= 1+4+4+3 = 12$$

$\therefore$ radius $= \sqrt{12}$

center $= (-1, 2, -2)$

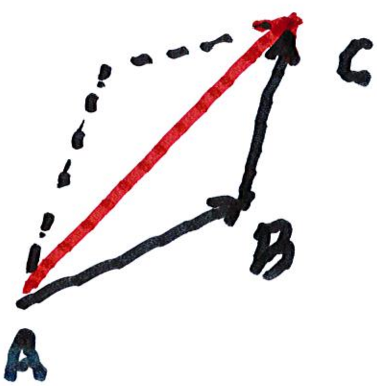
12.2 Vectors

Given points A and B, we can form a displacement vector $\overrightarrow{AB}$

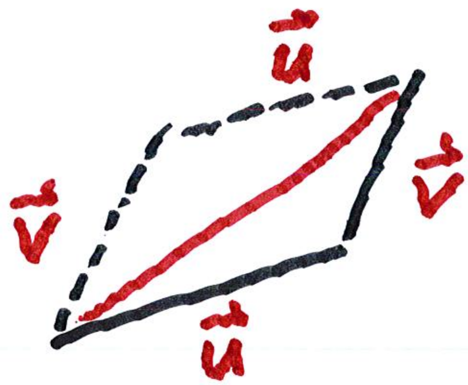
If $\overrightarrow{AB}$ is a translate of $\overrightarrow{CD}$, we consider $\overrightarrow{AB}$ and $\overrightarrow{CD}$ to be the same vector.

A is the initial point and B is the terminal point of $\overrightarrow{AB}$

Vector Addition is defined by the Parallelogram Law:

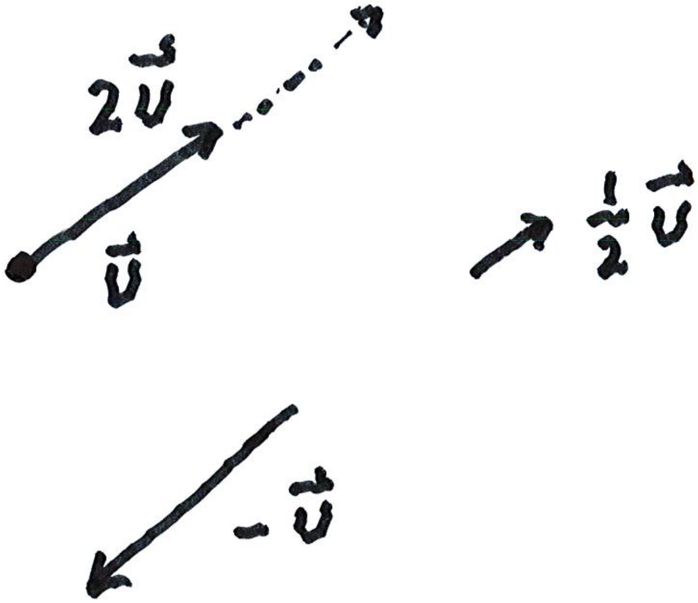


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$$\therefore\ \vec{u} + \vec{v} = \vec{v} + \vec{u}$$

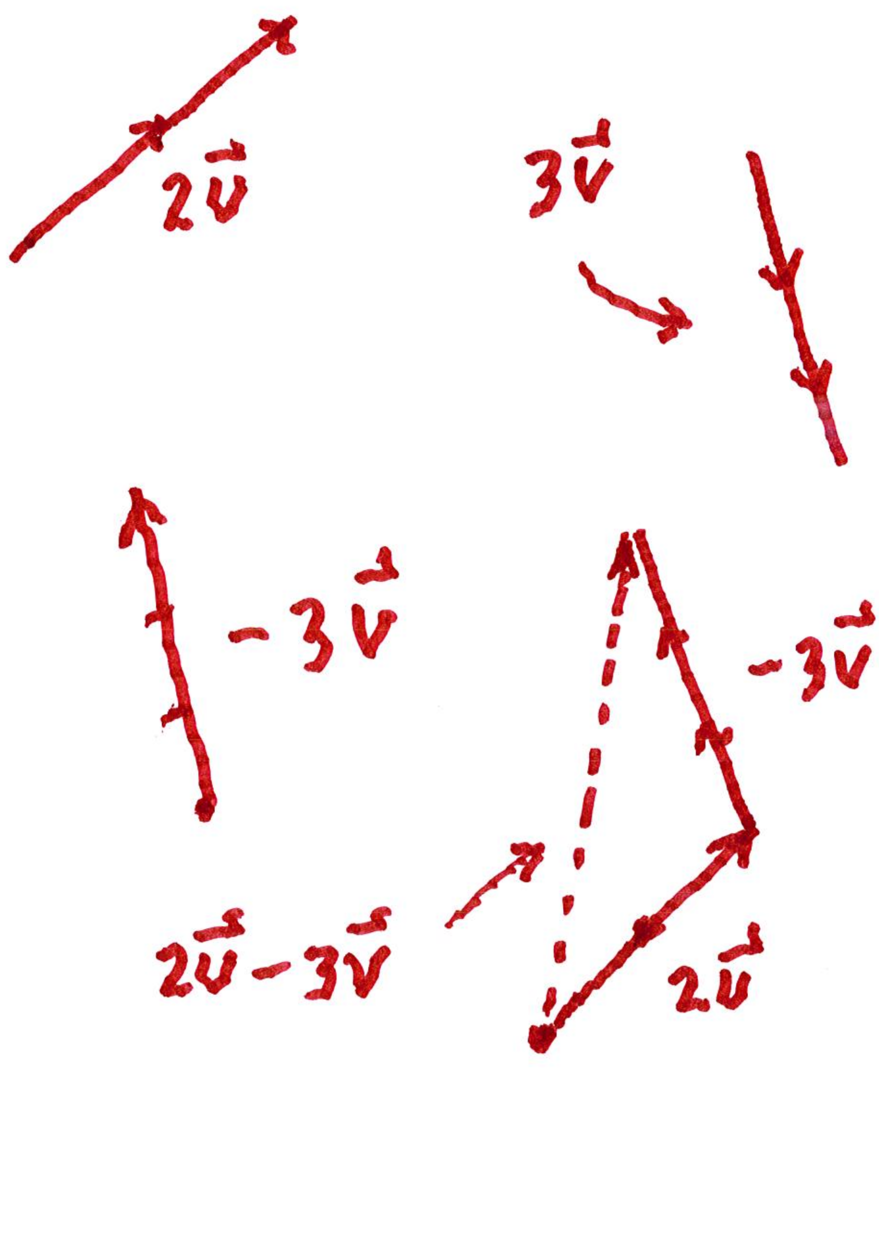
We can multiply $\vec{v}$ by any number c



If $\vec{u}$ is 

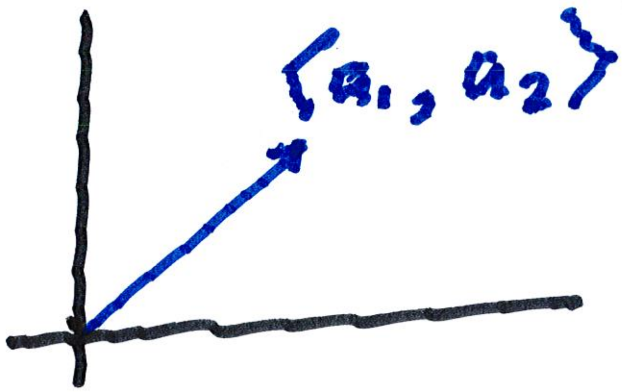
and $\vec{v}$ is  ,

what is $2\vec{u} - 3\vec{v}$



Components

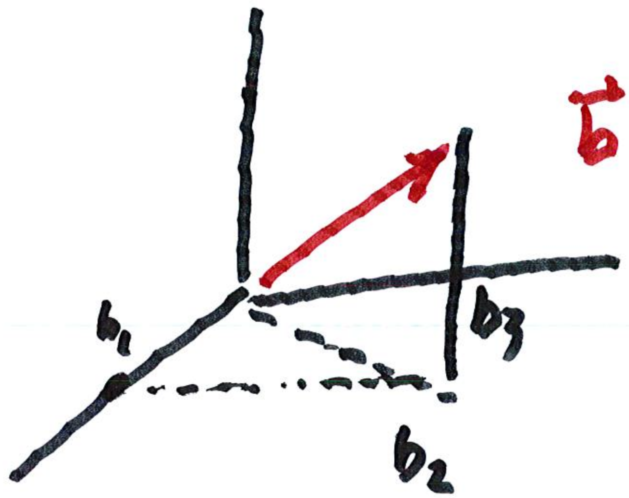
For vectors in $\mathbb{R}^2$:



$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$ absolute value

If $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$



$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a}$ and $\vec{b}$ are called position vectors.

In $\mathbb{R}^2$, $\vec{a} + \vec{b}$

$= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$

$= \langle a_1 + b_1, a_2 + b_2 \rangle$

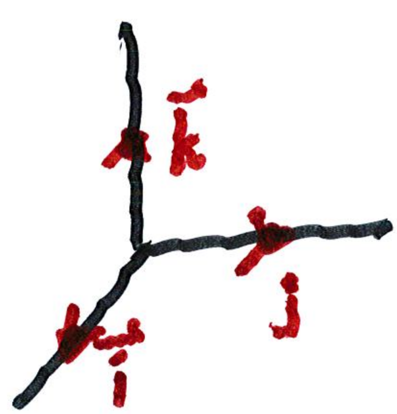
and $c\vec{a} = c\langle a_1, a_2\rangle$

$= \langle ca_1, ca_2\rangle$

Standard Basis Vectors

$\vec{i} = \langle 1, 0, 0\rangle \quad \vec{j} = \langle 0, 1, 0\rangle$

and $\vec{k} = \langle 0, 0, 1\rangle$



$\vec{v}$ is a unit vector if

$|\vec{v}| = 1.$

Ex. Find a vector $\vec{w}$ of length 3 that points in the opposite direction from

$\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$

$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$

$$\vec{u} = \frac{1}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

is a unit vector

$$\vec{v} = \frac{3}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

has length 3.

$$\vec{w} = \frac{-3}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

12.3 Dot Product

Suppose

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ and

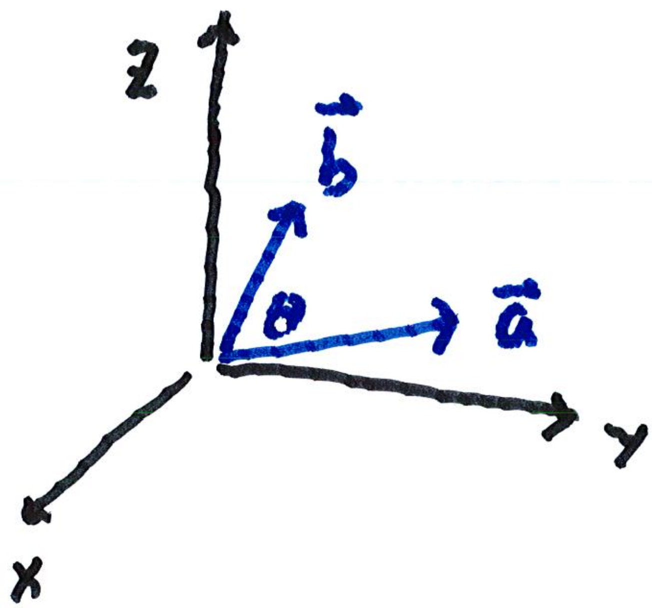
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of

$\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Geometric Meaning



Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \qquad (1)$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$

then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

and (1) is still true.

Ex. Find the angle θ between

$\vec{a} = \langle 3, -1 \rangle$ and $\vec{b} = \langle 2, 2 \rangle$

$\vec{a} \cdot \vec{b} = 3 \cdot 2 + (-1) \cdot 2 = 4$

$$|\vec{a}| = \sqrt{9+1} = \sqrt{10}$$

$$|\vec{b}| = \sqrt{4+4} = \sqrt{8}$$

$$\therefore \quad 4 = \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$= \sqrt{10}\sqrt{8} \cos\theta$$

$$\rightarrow \quad 4 = \sqrt{80} \cos\theta$$

$$4 = 4\sqrt{5} \cos\theta$$

$$\therefore \cos\theta = \frac{1}{\sqrt{5}} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

Suppose $\vec{a}$ and $\vec{b}$ satisfy

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow |\vec{a}||\vec{b}|\cos\theta = 0$$

$$\therefore \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$\therefore$ $\vec{a}$ and $\vec{b}$ <u>are perpendicular</u>

Conversely, if $\vec{a}$ and $\vec{b}$ are perpendicular, then $\underline{\vec{a}\cdot\vec{b} = 0}$

Ex. Show $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$

and $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

are perpendicular.

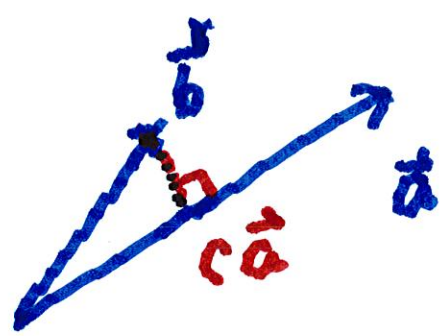
$\vec{a} \cdot \vec{b} = 2 \cdot 1 - 3 \cdot 1 + 1 \cdot 1 = 0$

$\Rightarrow \theta = \dfrac{\pi}{2}$

Scalar Projections

Find c so that

$\vec{b} - c\vec{a}$ is $\perp$ to $\vec{a}$



$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} = c$$

$$\left(\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2 \right)$$

$$\therefore \text{proj}_{\vec{a}} \vec{b} = c\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

This is the vector projection

of $\vec{b}$ onto $\vec{a}$

We can also write

$$c\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \text{comp}_{\vec{a}}\,\vec{b}$$

$\frac{\vec{a}}{|\vec{a}|}$ is a unit vector

$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ is the scalar projectso

It gives the size of $\text{proj}_{\vec{a}}\vec{b}$

in the direction of $\vec{a}$

Ex. Find $\text{comp}_{\vec{a}}\vec{b}$ if

$\vec{a} = 2\vec{i} + \vec{k}$

and $\vec{b} = 3\vec{i} + \vec{j} + 2\vec{k}$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{6+2}{\sqrt{4+1}} = \underline{\underline{\frac{8}{\sqrt{5}}}}$$

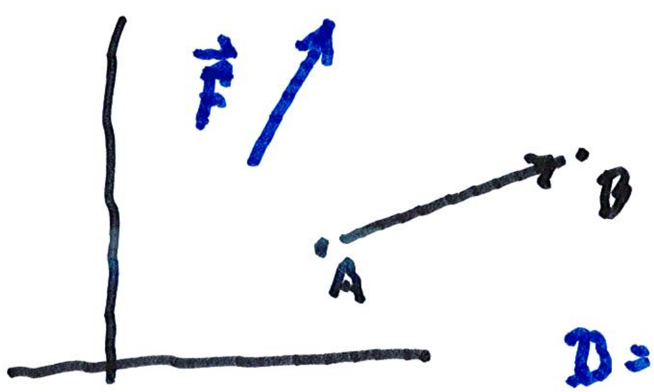
$$\parallel$$

$$\text{comp}_{\vec{a}}\vec{b}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{8}{5}\left(2\vec{i} + \vec{k}\right)$$

Work

Suppose that there is a constant force $\vec{F}$ and that a particle moves from A to B



$D = \overrightarrow{AB}$ = displacement vector.

The work done by the force is

$$W = \vec{F} \cdot \vec{D}$$