14.2 Limits and Continuity

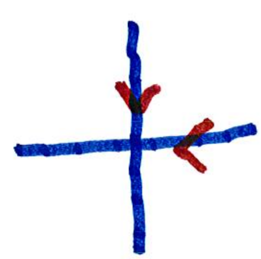
Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

What is $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Does the limit exist ?

Along the x-axis

$f(x,0) = 0$

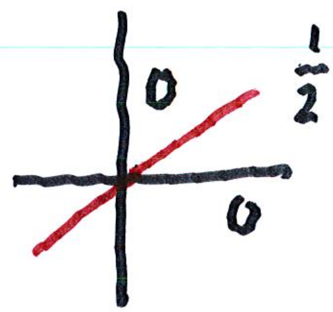


Along the y-axis,

$$f(0,y) = \frac{0\cdot y}{y^2} = 0$$

So, a good guess would be

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$$



Let's try

$y = x$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

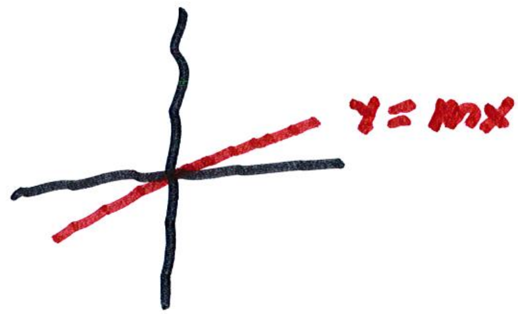
D. N. E.

The value of

~~What about~~ $\lim_{(x,y)\to(0,0)} f(x,y)$

should be the same for

all lines $y = mx$

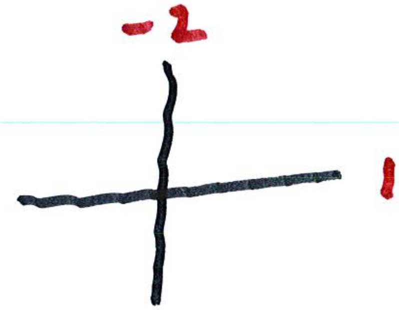


Ex. Find $\lim_{(x,y)\to(0,0)} \dfrac{x^2-2y^2}{x^2+y^2}$

On x-axis $\dfrac{x^2-0}{x^2} = 1$

On y-axis $\dfrac{0-2y^2}{0+y^2} = -2$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2} \quad \text{D. N. E.}$$



What is the correct definition?

Def'n. Suppose $f(x,y)$ is defined for all (x,y) near , (but not for (a,b).) We say that

Ex. Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Does $\lim f(x,y)$ exist as $(x,y) \to (0,0)$

Try $y = mx$.

$$f(x, mx) = \frac{x(mx)^2}{x^2+(mx)^4} =$$

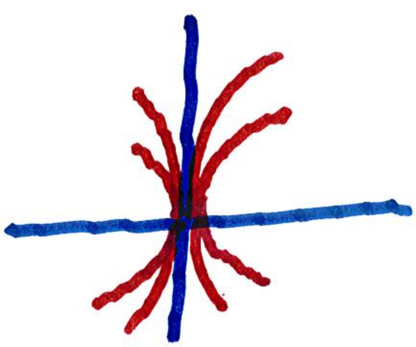
$$= \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^4},$$

so $f(x,y) \to 0$ on each line through $(0,0)$

But suppose $x = my^2$.

$$f(my^2, y) = \frac{my^2 \cdot y^2}{m^2y^4 + y^4}$$

$$= \frac{m}{m^2+1}$$



Note that $f(x,y)$ is a constant on each curve $x = my^2$.

In general, if $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

If the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b).

Consider $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

$$\left| \frac{3x^2y}{x^2+y^2} \right| \leq \frac{3(x^2+y^2)\cdot\sqrt{x^2+y^2}}{x^2+y^2}$$

$$\leq \frac{3(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2} = 3(x^2+y^2)^{\frac{1}{2}}$$

$$\to 0$$

as $(x,y) \to (0,0)$

Using the above intuitive definition, one can show

(i) $\lim\limits_{(x,y)\to(a,b)} x = a$

(ii) $\lim\limits_{(x,y)\to(a,b)} y = b$

(iii) $\lim\limits_{(x,y)\to(a,b)} c = c$

Continuity

We say a function of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

Using the usual properties of limits, one can see that sums, differences and products and quotients of continuous functions are also continuous,

(provided the denominator is $\neq 0$.)

A polynomial is a sum of terms of the form $cx^m y^n$

For ex, $f(x,y) = x^3 - 2xy + y^2 - 4$ is a polynomial, and therefore continuous everywhere. A rational function is a quotient of polynomials

$$g(x,y) = \frac{xy^2 - y^3}{x^4 + y^2 - 3}$$

Ex. Where is the function defined?

$$\frac{2xy^2}{3x^2+y^4}$$

(only at (0,0)

It's continuous at each point (a,b) if $(a,b) \neq (0,0)$.

But $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ $\underline{\underline{\text{is}}}$

continuous at $(0,0)$ if we set $f(0,0) = 0$.

A function $f(x, y, z)$

has a limit at (a, b, c)

if $f(x,y,z)$ approaches the

same number L, no matter

how $(x, y, z) \to (a, b, c)$.

We say f is continuous at

(a, b, c) if $\displaystyle\lim_{(x,y,z)\to(a,b,c)} f(x, y, z)$

$= f(a, b, c)$

11. $\lim \dfrac{y^2 \sin^2 x}{x^4 + y^4}$

$\sin x \sim x \rightarrow \sin^2 x \approx x^2$

13 $\lim \dfrac{xy}{\sqrt{x^2+y^2}}$

$\cancel{|xy| \leq \sqrt{\ }}$ $|xy| \leq x^2 + y^2$

$\dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$

17. $$\lim \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

$$\sqrt{1+u} \approx 1+\frac{1}{2}u+\cdots$$

$$\sqrt{1+x^2+y^2} \sim 1+\frac{1}{2}(x^2+y^2)+\cdots$$

$$\therefore \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \sim \frac{x^2+y^2}{\frac{1}{2}(x^2+y^2)}+\cdots$$

$$= 2+\cdots$$

14. $$\lim \frac{x^4 - y^4}{x^2 + y^2}$$

$$= \lim \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2}$$

$$= \lim x^2 - y^2 = 0$$ as $(x,y) \to (0,0)$

16. $$\lim \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

$\sin y \sim y$ if y is small.

$$f(x,y) \sim \cancel{x^2 y \sin} \frac{x^2 y^2}{x^2 + 2y^2} \to 0$$

Def.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$,

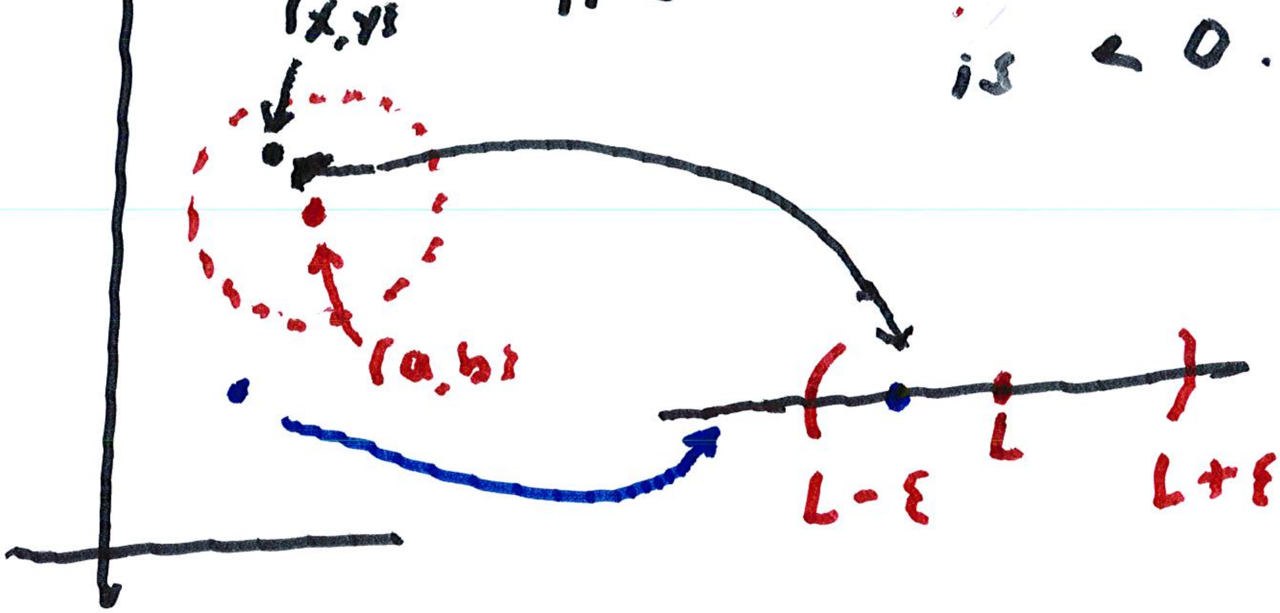
there is a corresponding

number $\delta > 0$, so that <u>if</u>

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

then $|f(x,y) - L| < \varepsilon$

if $0 < \sqrt{(x-a)^2 + (y-b)^2}$ is $< \delta$.



You have to show that if

(x, y) is within δ of (a, b),

then $f(x, y)$ is within ε of L.