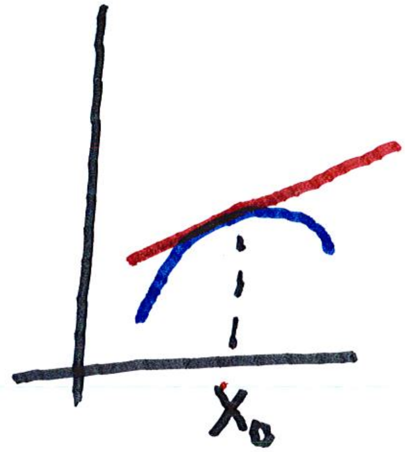
14.4 Tangent Planes and Approximations

Suppose we are given a curve $y = f(x)$. From 1-variable calculus, if a curve $y = f(x)$ passes through (x_0, y_0), then the line that best approximates the curve is

$$y - y_0 = f'(x_0)(x - x_0)$$



Now suppose are given a surface

$z = f(x, y)$ that passes through (x_0, y_0, z_0). We want to know which plane best approximates the surface.

A plane can be expressed as

(1) $$z - z_0 = a(x - x_0) + b(y - y_0)$$

If we fix $y = y_0$ and allow x to vary, then the curve becomes $z - z_0 = f(x, y_0)$

and the slope of the line is $\dfrac{\partial f}{\partial x}(x_0, y_0)$

If we fix $x = x_0$ and allow y to vary, then the curve becomes $z - z_0 = f(x_0, y)$,

and the slope of the line is $\frac{\partial f}{\partial y}(x_0, y_0)$. It makes sense that the coefficients $\underline{a}$ and $\underline{b}$ in (1) are $a = \frac{\partial f}{\partial x}(x_0, y_0)$

and $b = \frac{\partial f}{\partial y}(x_0, y_0)$

Hence the plane that

best fits the surface $z = f(x, y)$

is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Since $z_0 = f(x_0, y_0)$ we get

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Ex. Find the plane that

approximates $z = \underbrace{x^3 - xy^2 + y^3}_{f(x,y)}$

at $(1, 2, 5)$.

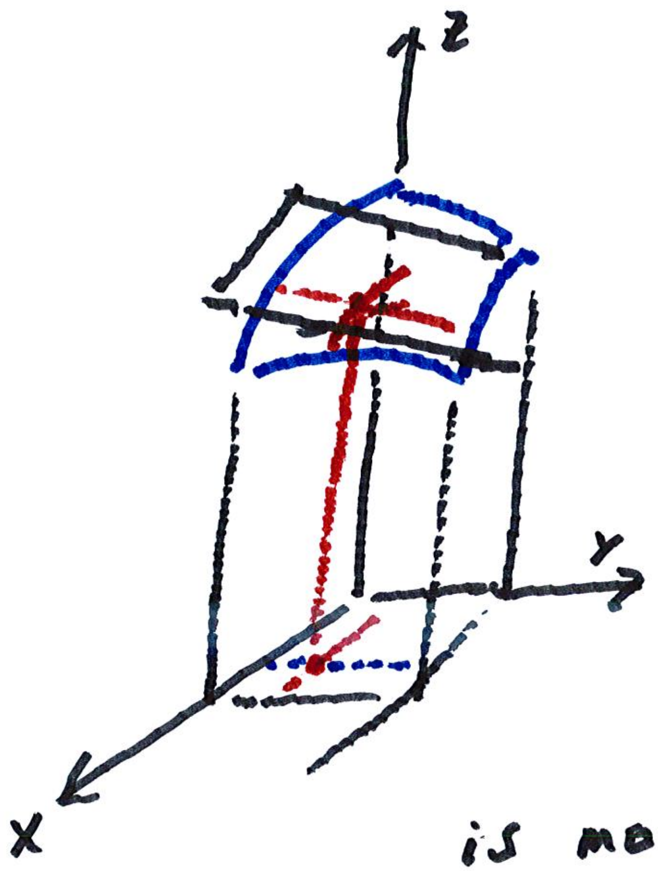
$$\frac{\partial f}{\partial x}(1,2) = 3x^2 - y^2 = 3 - 4 = -1$$

$$\frac{\partial f}{\partial y}(1,2) = -2xy + 3y^2 = -4 + 12 = 8$$

$\rightarrow$ $z - 5 = -1(x-1) + 8(y-2)$

One can also rewrite this as

$$z = -x + 8y - 10$$



As (x, y) gets closer to (x_0, y_0) and we zoom in the surface is more like the plane.

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the rate of

change in the x-direction

and

$\frac{\partial f}{\partial y}(x_0, y_0)$ is the rate of

change in the y-direction.

Find the equation of the <u>plane</u>

that approximates

$z = 3x^2 - 2xy^3 + y^3$ at $(2, 1)$

$$\frac{\partial f}{\partial x} = 6x - 2y^3 \quad = 12 - 2 = \underline{10}$$

$$\frac{\partial f}{\partial y} = -6xy^2 + 3y^2 = -12 + 3 = \underline{-9}$$

Also, $f(2,1) = 12 - 4 + 1 = 9$

$$z - 9 = 10(x-2) - 9(y-1)$$

This plane is called the tangent plane at $(2,1,9)$

In general the tangent plane is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

This is called the linear approximation or the tangent approximation of

f at (x_0, y_0). This is a

good approximation if

$\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$

are continuous near (x_0, y_0).

If those functions are <u>not</u>

continuous, it may be a poor

approximation.

Ex. $f(x,y) = \dfrac{xy}{x^2+y^2}$, $f(0,0) = 0$

We can use the approximation

to measure changes in Δf

$$f(x,y) - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a)$$

$$+ \frac{\partial f}{\partial y}(a,b)(y-b)$$

For a differentiable function $f(x,y)$ we define the differentials dx and dy to be independent variables. Then the total differential is defined by

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

The total differential means that if dx and dy are small changes in x and y, then the change in z is described as above.

Thus, if f is differentiable at (a,b), then a good approximation of $f(x,y)$ near (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

If $z = f(x,y)$, we define the increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b).$$

Def'n. If $z = f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed as

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

Thm. If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Ex. Show $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$, and find its linearization to approximate $f(1.1, -0.2)$

$f_x = e^{xy} + xye^{xy} \qquad f_y = x^2 e^{xy}$

f_x and f_y are both continuous functions, so f is differentiable

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y$$

$$= 1 + 1(x-1) + (y-0) = x+y$$

$$= 1 + (.1) + (-0.2) = .9$$

This is the linearization or the linear approximation of $f(x,y) = xe^{xy}$ at $(1,0)$

If $z = f(x,y)$, we sometimes write

$$dz = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$

Ex. If $f(x,y) = x^2 + 3xy - y^2$, find dz

$f_x = 2x + 3y \qquad f_y = 3x - 2y$

$\therefore df = (2x+3y)\,dx + (3x-2y)\,dy$

Put $x = 3$, $y = 2$,

and $\Delta x = .1$ and $\Delta y = .2$

$(2x+3y) = 6+6 = 12$

$(3x-2y) = 9-4 = 5$

$\therefore \Delta z = 12(.1) + 5(.2) = 2.2$

For functions of 3 variables:

$$f(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Ex. Find the first partial derivatives of

$$f(x,y,z) = xz - 5x^2y^3z^4$$

$$f_x = z - 10xy\,z^4$$

$$f_y = -15x^2y^2z^4$$

$$f_z = x - 20x^2y^3z^3$$

Ex. Find all second derivatives

of $f(x,y) = x^3y^5 + 2x^4y$

$f_x = 3x^2y^5 + 8x^3y$

$f_y = 5x^3y^4 + 2x^4$

$f_{xx} = 6xy^5 + 24x^2y$

$f_{xy} = 15x^2y^4 + 8x^3$

$f_{yy} = 20x^3y^3$

$f_{xy} = f_{yx} = 15x^2y^4 + 8x^3$