14.7 Maximum and Minimum Values.

Def'n. A function $f(x, y)$ has a local maximum at (a, b) if $\underline{f(x,y) \le f(a,b)}$ when (x,y) is near (a,b).

The number $f(a,b)$ is called the <u>local maximum value</u>.

Similarly, if $f(a,b) \le f(x,y)$ for (x,y) near (a,b), then f has a local minimum, and $f(a,b)$ is a local minimum value.

Theorem. If f has a local minimum or local maximum at (a,b) and the first-order partial derivatives exist at (a, b), then

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Ex. If we set $g(x) = f(x,b)$, and if f has a local maximum or minimum at (a,b), then g has a local maximum (or minimum) at a, so $f_x(a,b) = 0$. Also, if we set $G(y) = f(a, y)$, then

$G'(b) = 0$, which implies

that $f_y(a, b) = 0$.

A point (a, b) is a critical

point of f if

$f_x(a, b) = 0$ and $f_y(a, b) = 0$.

At a critical point, then f can have a local maximum or local minimum or neither.

Ex. Let $f(x, y) = x^2 - 4x + 6y + y^2 + 2$.

Find all crit. pts.

then $f_x = 2x - 4$

$f_y = 6 + 2y$

If $f_x = 0$, then $x = 2$, and

if $f_y = 0$, then $y = -3$

If we complete the square.

$f(x, y) = (x-2)^2 + (y+3)^2 - 11.$

$\therefore$ f has a local minimum at $(2, -3)$. In fact f has an absolute minimum at $(2, -3)$.

because $f(x, y) \geq f(2, -3)$ for all (x, y) in $\mathbb{R}$.

Ex. Find all extreme values

of $f(x, y) = x^2 - y^2$.

$f_x(x, y) = 2x$

$f_y(x, y) = -2y$.

$\therefore$ Only critical point

is $(0, 0)$.

$f(x, 0) = x^2$, so $f(x, 0)$

takes on all positive values.

Also, $f(0, y) = -y^2$ takes

on all negative values.

$\therefore$ f has NO extreme values

Note $z = x^2 - y^2$ has

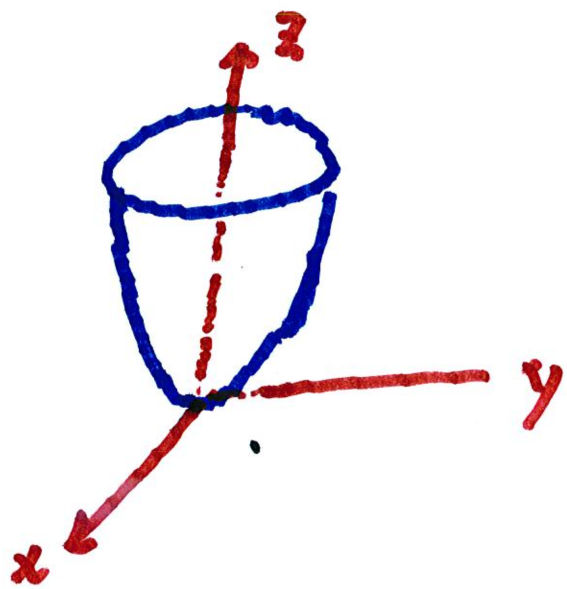
a saddle at $(0,0)$

On the other hand,

$z = x^2 + y^2$ is a paraboloid

and has a global minimum

at $(0, 0)$



In 1-variable calculus,

$y = f(x)$ has a local minimum at x_0

if $f''(x_0) > 0$.

For a function $f(x,y)$ we have:

Second Derivative Test.

Suppose the second partial derivatives of are continous on a disk about (a,b), and suppose that (a,b) is a critical point of f. Let

The 2nd. Derivatives Test

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

~~Local min~~

(a) If $D > 0$ and $f_{xx} > 0$,

then $f(a,b)$ is a local minimum

(b) If $D > 0$ and $f_{xx}(a,b) < 0$,

then f has a local maximum $f(a,b)$

(c) If $D < 0$, then $f(a,b)$

is not a local maximum or minimum

Note 1 If $D < 0$, then one can show that the graph of f has a saddle point

Note 2 If $D = 0$

then the test gives no information, i.e. f could have a local min. or max or (a, b) could be a saddle point.

Ex. Find all local maximum and local minimum values and saddle points:

Ex. $f(x,y) = xy(1-x-y)$

$$= xy - x^2y - xy^2$$

$f_x = y - 2xy - y^2 = y(1-2x-y)$

$f_y = x - x^2 - 2xy = x(1-2y-x)$

Note $f_x = 0$ if $y = 0$ or $2x+y = 1$

and $f_y = 0$ if $x = 0$ or $x + 2y = 1$

Pts. at $(0,0)$,

or ~~y~~ $(0,1)$

or $(1,0)$

or $\left.\begin{array}{l} x+2y=1 \\ 2x+y=1 \end{array}\right\}$ $x=\frac{1}{3}$, $y=\frac{1}{3}$

$\left(\frac{1}{3}, \frac{1}{3}\right)$.

Calculate D:

$f_{xx} = -2y \quad f_{yy} = -2x$

$f_{xy} = 1 - 2x - 2y$

$D = 4xy - (1-2x-2y)^2$

At $(0,0)$ $D = 0 - 1 = -1$

saddle pt

At $(1,0)$ $D = 0 - (1-2)^2 = -1$

saddle pt

At $(0,1)$ $D = 0 - 1 = -1$

saddle pt

At $\left(\frac{1}{3}, \frac{1}{3}\right)$

$$D = \frac{4}{9} - \left(1 - \frac{4}{3}\right)^2 = \frac{4}{9} - \frac{1}{9}$$

$$= \frac{1}{3} > 0$$

$f_{xx} = -2 \cdot \frac{1}{3} < 0$

$\therefore$ f has a local maximum at $\left(\frac{1}{3}, \frac{1}{3}\right)$

Ex. $f(x,y) = 3x - x^3 - 2y^2 + y^4$

1. Find critical pts

$f_x : 3 - 3x^2 = 0 \rightarrow x = \pm 1$

$f_y = -4y + 4y^3 = 4y(-1 + y^2)$

$y = 0, 1, \text{ or } -1$

$\therefore (1,-1) \quad (-1,-1)$

$(1, 0) \quad (-1, 0)$

$(1, 1) \quad (-1, 1)$

$f_{xx} = -6x$

$f_{yy} = (-4 + 12y^2)$

$f_{xy} = 0$

$\therefore\ D = 24x - 72xy^2$

1. At $(1, -1)$, $D = 24 - 72 = -48$

SADDLE PT

2. At $(1, 0)$ $D = 24$

and $f_{xx} = -6$
< 0

$\rightarrow$ local max at $(1, 0)$

3 At $(1, 1)$,

$\rightarrow D = -48 \longrightarrow$ $\therefore$ saddle point

4. At $(-1,-1)$

$\rightarrow D = -48 \rightarrow$ saddle pt

$f_{xx} = +2y = -2 < 0 \therefore$ Local Max

5. At $(-1, 0)$

$D = -24 \rightarrow$ saddle point

6. At $(-1, 1)$

$D = 48 \qquad f_{xx} = -6x = 6$

$\therefore$ local minimum

Ex. Set $f(x,y) = (x-y)(1-xy)$

$$f(x,y) = x - y - x^2 y + xy^2$$

$f_x = 1 - 2xy + y^2 = 0$

$f_y = -1 - x^2 + 2xy = 0$

Add. $y^2 - x^2 = 0$

$y = x \rightarrow x^2 - 2x^2 + 1 = 0$

$\rightarrow x = \pm 1 \quad (1,1)$ or $(-1,1)$

If $y = -x$.

$\rightarrow 1 + 2x^2 + x^2 = 0$

$\rightarrow (x+1)^2 = 0 \rightarrow x = -1$

$\therefore (x, y) = (-1, 1)$

$\therefore$ 3 points

$(1, 1)$, $(-1, -1)$, $(-1, 1)$

$f_{xx} = -2y \quad f_{yy} = 2x$

$f_{xy} = -2x + 2y$

$\therefore D = -4xy + 4(y-x)^2$

$= 4x^2 - 12xy + 4y^2$

1. $(1,1)$ $D = -4 \rightarrow$ saddle point

$f(1,1) = 0$

2. $(-1,-1)$ $D = 4 - 12 + 4 = -4$

saddle pt.

$f(-1,1) = 0$

3. $(-1,1)$ $D = 20$ $f_{xx} = -2$

$\therefore$ local maximum.