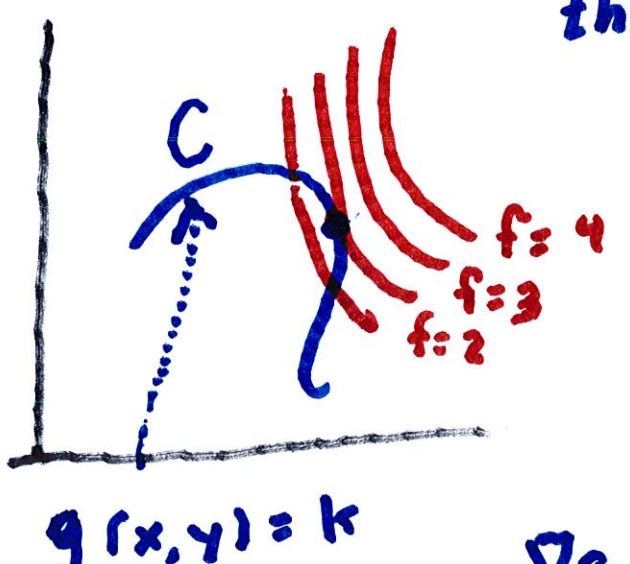
14.8 Lagrange Multipliers

Suppose C is a curve defined by $g(x,y)=k$, where $g(x,y)$ is a differentiable function. We want to define the maximum and minimum of $f(x,y)$ on C.

Consider the picture. It suggests



that if f has an extreme value on C, then ∇f should be a multiple of ∇g, i.e. $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$

Thus, to find the extreme values of f, we must find a point (x_0, y_0) in C so that

$$g(x_0, y_0) = k, \qquad \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Thus we need to solve 3 equations

$$\frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y)$$

$$\frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y)$$

$$g(x,y) = k.$$

The unknown variables are x, y, and λ

The equation $g(x,y) = k$ is called a constraint.

We can use the same method to find the extreme values of $f(x,y,z)$ on the surface $g(x,y,z) = k$.

Lagrange Multipliers: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, we need to find all values of

x, y, z and λ such that

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

and $g(x, y, z) = k$.

Thus, find all x, y, z and λ

so that

$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z$

and $g(x, y, z) = k$

Ex. Find the maximum and minimum of $f(x,y) = x^2 - 2x + 2y^2$ on the line $4x - 6y = 1$

Thus, $g(x,y) = 4x - 6y = 1$

$\nabla f = \lambda \nabla g$, $\langle 2x-2, 4y \rangle = \lambda \langle 4, -6 \rangle$

$f_x = \lambda g_x \quad 2x - 2 = 4\lambda$

$f_y = \lambda g_y \quad 4y = -6\lambda$

$g(x,y) = k \quad 4x - 6y = 1$

$\therefore \quad 2x - 2 = \lambda \cdot 4$

$4y = \lambda \cdot (-6)$

and $\quad 4x - 6y = 1$

$\rightarrow \quad x = 2\lambda + 1$

$y = -\frac{3}{2}\lambda$

Sub into g-equation

$4(2\lambda + 1) - 6\left(-\frac{3}{2}\lambda\right) = 1$

$\rightarrow 17\lambda = -3 \rightarrow \lambda = \frac{-3}{17}$

Plug this into the equations

for x and y:

$$x = -\frac{6}{17} + 1 = \frac{11}{17}$$

$$y = -\frac{3}{2} \cdot \frac{(-3)}{17} = \frac{9}{34}$$

$$\therefore f\left(\frac{11}{17}, \frac{9}{34}\right)$$

$$= \frac{11^2}{17^2} - \frac{22}{17} + \frac{2 \cdot 81}{34^2}$$

Absolute Minimum.

This works in any dimensions

Ex. Find the maximum any

minimum of $f(x,y,z) = x + 2y + 4z$

on the surface $x^2 + y^2 + z^2 = 1$

$$\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 1$$

Note that $\lambda \neq 0$.

$1 = 2\lambda x$

$2 = 2\lambda y$

$4 = 2\lambda z$

$x^2 + y^2 + z^2 = 1$

Solve for x, y, and z

in terms of λ:

$$x = \frac{1}{2\lambda}$$

$$y = \frac{1}{\lambda}$$

$$z = \frac{2}{\lambda}$$, and $x^2 + y^2 + z^2 = 1$

$$\therefore \quad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 = 1$$

$$\frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{16}{4\lambda^2} = 1$$

$$\frac{21}{4\lambda^2} = 1 \rightarrow 4\lambda^2 = 21$$

$$\rightarrow \lambda = \frac{\pm\sqrt{21}}{2}$$

$$\lambda = +\frac{\sqrt{21}}{2} \rightarrow x = \frac{1}{2} \cdot \frac{2}{\sqrt{21}} = \frac{1}{\sqrt{21}}$$

$$y = \frac{2}{\sqrt{21}}$$

$$z = \frac{4}{\sqrt{21}}$$

$$\left(\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}} \right)$$

Note: This gives max value $= \sqrt{21}$

If $\lambda = \frac{-\sqrt{21}}{2}$, then

$$\left(\frac{-1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-4}{\sqrt{21}} \right)$$

Ex. Find the min and max of

$f(x,y) = e^{xy}$ on the

curve $x^3 + y^3 = 16$

$f(x,y)$

$g(x,y)$

$$\nabla f = \lambda \nabla g, \quad g(x,y) = k$$

$$\nabla(e^{xy}) = \lambda(\nabla g) \quad \nabla g = \langle 3x^2, 3y^2 \rangle$$

$$f_x = ye^{xy} = \lambda \cdot 3x^2 \quad f_y = xe^{xy} = \lambda \cdot 3y^2$$

$$\lambda = \frac{ye^{xy}}{3x^2} \qquad \lambda = \frac{xe^{xy}}{3y^2}$$

$$\therefore \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$$

$$\rightarrow 3y^3 = 3x^3$$

$$y^3 = x^3 \rightarrow (y-x)\underbrace{(y^2 + yx + x^2)}_{>0} = 0$$

$\therefore y = x$ sub into g - eq'n:

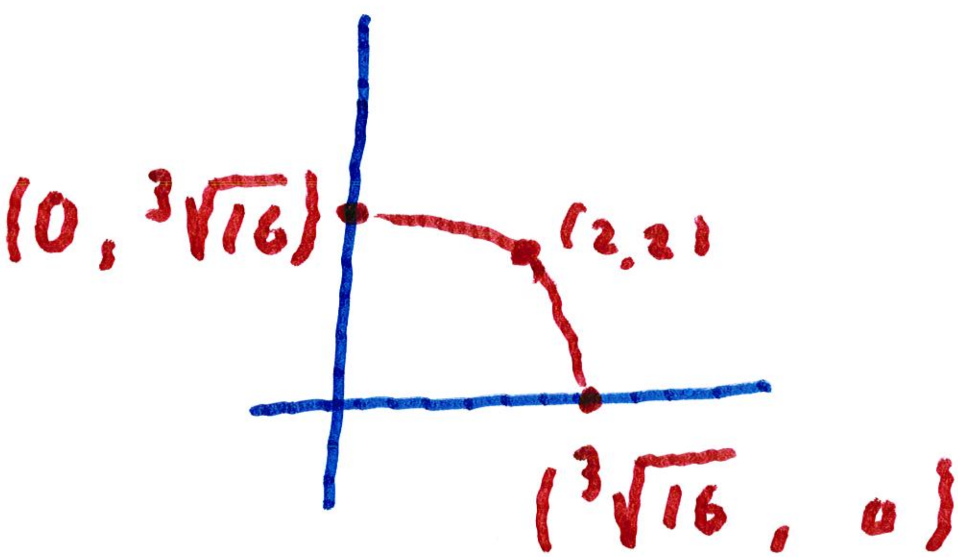
$2x^3 = 16$

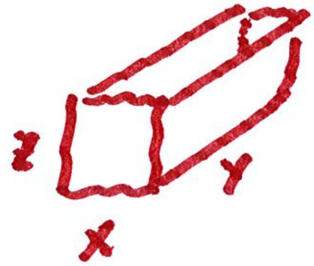
$\rightarrow x^3 = 8 \rightarrow x = 2$

$\therefore y = x \rightarrow y = 2$

$\therefore$ crit. pt is $(2, 2)$

on $x^3 + y^3 = 16$





Total area $= xy + 2xz + 2yz$

$= 12$

Find x, y, z that maximize $V = \underbrace{xyz}_{f(x,y,z)}$

$g \quad xy + 2xz + 2yz = 12$ $\quad \nabla g = \langle y+2z,\ x+2z,\ 2x+2y \rangle$

$yz = V_x = \lambda(y+2z)$

$xz = V_y = \lambda(x+2z)$

$xy = V_z = \lambda(\cancel{1y+2z}) = \lambda(2x+2y)$

Multiply by x, y, z resp.

$$xyz = \lambda(xy + 2xz)$$

$$xyz = \lambda(2yz + xy)$$

$$xyz = \lambda(2xz + 2yz)$$

$\lambda \neq 0$, because that

would imply $yz = xz = xy = 0$

which would contradict g-equation

So we get

$$2xz + xy = 2yz + xy$$

$$\rightarrow \quad 2xz = 2yz \quad \rightarrow \quad \underline{\underline{y = x}}$$

But then, we can conclude from Eq'n 2 and Eq'n 3

that $y = 2z$.

Hence we have $x = y = 2z$

$$\Rightarrow \quad 4z^2 + 4z^2 + 4z^2 = 12$$

Therefore $z=1$, which means that the dimensions are

$(2, 2, 1)$.

Ex. Find max and min of

$$\underbrace{x^2+y+z}_{f} \quad \text{on} \quad \underbrace{x^2+y^2+z^2}_{g}=1$$

$$2x = 2\lambda x$$

$$\left.\begin{aligned} 1 &= 2\lambda y \\ 1 &= 2\lambda z \end{aligned}\right\} \text{Note } \lambda \neq 0$$

$\to y = z$

1st Eq'n $\to$ $x=0$ or $\lambda = 1$

If $x=0 \to 2y^2=1$

$\to y = \frac{1}{\sqrt{2}}, \; z = \frac{1}{\sqrt{2}}$

or $y = -\frac{1}{\sqrt{2}}, \; z = -\frac{1}{\sqrt{2}}$

If $\lambda = 1$, then Eq'n 2 and 3

$\rightarrow y = \frac{1}{2}, \quad z = \frac{1}{2}$

There are four points

$\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right)$ $\quad \lambda = 1 \quad f = \frac{3}{2}$

and $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ and $\left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$

$\lambda = \frac{1}{\sqrt{2}} \qquad \lambda = -\frac{1}{\sqrt{2}}$

$f = \sqrt{2} \qquad f = -\sqrt{2}$

$\therefore$ Min $= -\sqrt{2}$, Max $= \frac{3}{2}$

Ex. Find the max and min of $2x-3y-1$ if (x,y) lies on the circle $x^2+y^2=1$

$f(x,y) = 2x-3y-1$, $g(x,y) =$
$x^2+y^2=1$

$\nabla f = \langle 2, -3 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\therefore \quad 2 = 2\lambda x, \quad -3 = 2\lambda y$

$\rightarrow \quad x = \frac{1}{\lambda} \qquad y = \frac{-3}{2\lambda}$

$\lambda \neq 0$

$$\left(\frac{1}{\lambda}\right)^2 + \left(\frac{-3}{2\lambda}\right)^2 = 1$$

$$\frac{1}{\lambda^2}\left(1 + \frac{9}{4}\right) = 1$$

$$\lambda^2 = \frac{13}{4}$$

$$\lambda = \frac{\pm\sqrt{13}}{2}$$

$$\therefore \quad x = \frac{2}{\sqrt{13}} \qquad y = -\frac{3}{2}\cdot\frac{2}{\sqrt{13}}$$

$$(x, y) = \frac{1}{\sqrt{13}}(2, -3)$$