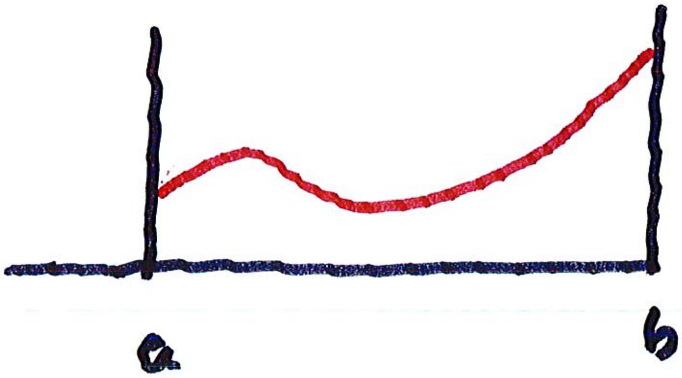
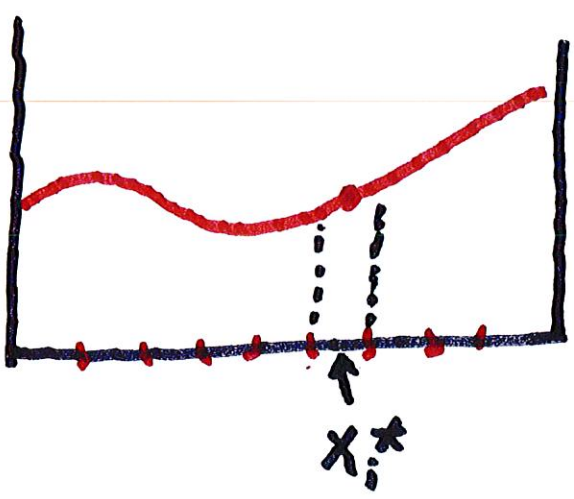
15.1 Double Integrals over Rectangles.

Remember, the integral in ~~in~~ one variable is essentially the area under a curve {when $f(x) \geq 0$} between $x=a$ and $x=b$.



To compute the area, we decompose $[a, b]$ into n small intervals: $[x_{i-1}, x_i]$ for $1 \leq i \leq n$ and $x_i - x_{i-1} \leq \frac{b-a}{n}$



We let x_i^* be any random point

in $[x_{i-1}, x_i]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{n=1}^{\infty} f(x_i^*)\,\Delta x,$$

where $\Delta = \dfrac{b-a}{n}$

Suppose that $f(x, y)$ is defined

for $\begin{cases} a \le x \le b & \text{and} \\ c \le y \le d \end{cases}$

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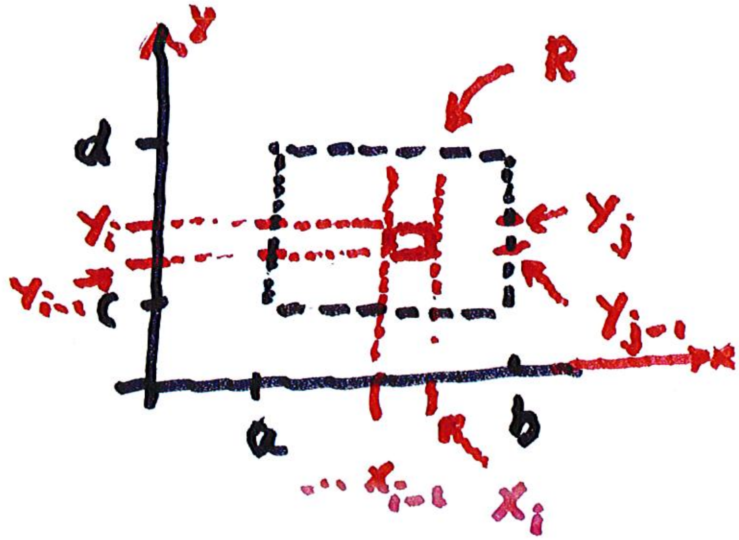


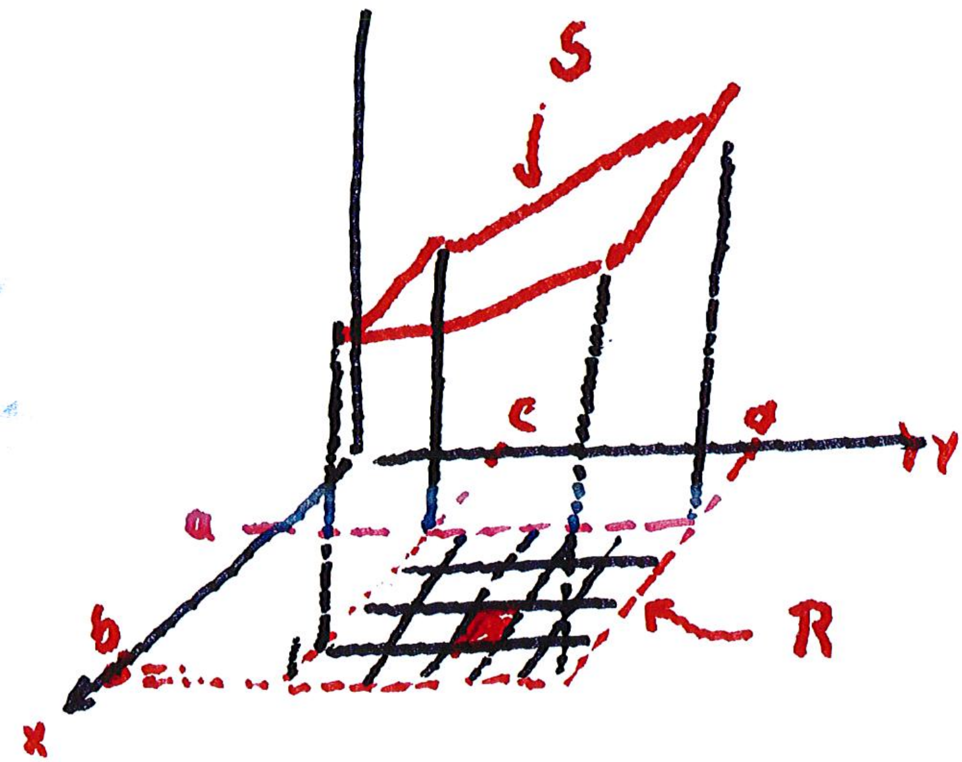
The small rectangle is defined by

$x_{i-1} \le x \le x_i$ and $y_{j-1} \le y \le y_j$

$\Delta x = \dfrac{b-a}{m}$ $\Delta y = \dfrac{d-c}{n}$

The small rectangle is

denoted by R_{ij}.



We want to compute the

volume under S, $z = f(x, y)$

$$\text{with} \left\{ \begin{array}{c} a \le x \le b \\ c \le y \le d \end{array} \right\} = R$$

Let $\{x_{ij}^*, y_{ij}^*\}$ be a randomly selected point in R_{ij}

The volume above R_{ij} and below S is

$$\approx f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$$

and the total volume is

approximately

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$$

If we let $m \to \infty$ and $n \to \infty$

then if $f(x,y)$ is continuous

at all (x,y) except for

" a small set ", then

the limiting value of the above

double sum is

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where $\Delta A = \Delta x \cdot \Delta y$

Note that the above definition

of

$$\iint_R f(x,y)\,dA$$ makes sense

when $f(x,y)$

is ≥ 0 or ≤ 0.

Sometimes, instead of (x_{ij}^*, y_{ij}^*)

one uses (x_i, y_j) instead of

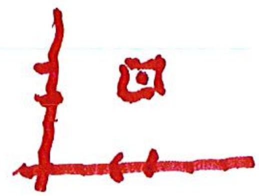
(x_{ij}^*, y_{ij}^*) Upper Right Corner

or $(\bar{x}_i, \bar{y}_j)$, where

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2}$$

and

$$\bar{y}_j = \frac{y_{j-1} + y_j}{2}.$$



15.2 Iterated Integrals

How do we actually compute

$$\iint_R f(x,y)\, dA \ ?$$

Note that $dA = dy\, dx$

$$\Rightarrow \int_{\substack{a \\ x}}^{b} \int_{\substack{c \\ y}}^{d} f(x,y)\, dy\, dx$$

$$= \int_{a}^{b} \left[\underbrace{\int_{c}^{d} f(x,y)\, dy} \right] dx$$

$$= \int_{a}^{b} A(x)\, dx \,, \quad \text{where for}$$

fixed x, $A(x) = \int_{c}^{d} f(x,y)\, dy$

Note, we work from the inside out:

$$\int_{\substack{a\\x}}^{b}\left(\int_{\substack{c\\y}}^{d} f(x,y)\,dy\right)dx$$

$$= \int_{a}^{b}\left[\int_{c}^{d} f(x,y)\,dy\right]dx$$

Compute $\int_0^3 \int_0^1 x^3 y \, dy \, dx$

$0 \le x \le 3 \qquad 0 \le y \le 1$

First integrate $\int x^3 y \, dy$ from inside

$$\cancel{\int_0^3 \frac{x^3 y^2}{2} \Big|}$$

$$= \int_{\substack{0 \\ x}}^3 \int_{\substack{0 \\ y}}^1 x^3 y \, dy \, dx$$

$$= \int_0^3 \frac{x^3 y^2}{2} \Big|_{0=y}^{1=y} dx$$

$$= \int_0^3 \frac{x^3}{2}\, dx = \frac{x^4}{8}\Big|_0^3$$

$$= \frac{3^4}{8} = \frac{81}{8}$$

Now compute

$$\int_0^1 \int_0^3 x^3 y \, dx \, dy$$

$$= \int_0^1 \frac{x^4}{4} \cdot y \, \Big|_0^3 \quad = \frac{3^4}{4} \, \frac{y^2}{2}$$

$\frac{81}{8}$ "

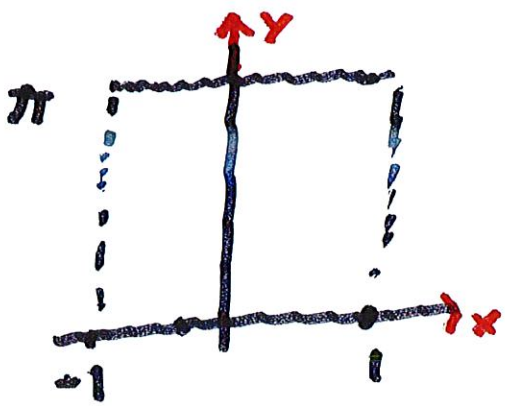
We get the same number:

Fubini's Thm. If $f(x,y)$ is continuous on $\underbrace{\left\{\begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}\right\}}_{R}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Find the volume of the solid enclosed by $z = e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$



$$= \int_{-1}^{1} \int_{0}^{\pi} e^x \sin y \, dy \, dx$$

x y

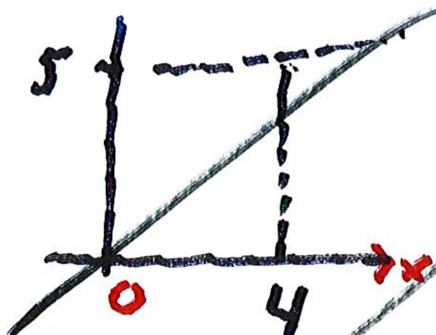
$$= \int_{-1}^{1} -e^x \cos y \Big|_0^{\pi} dx$$

$$= -\int_{-1}^{1} e^x (-1 - 1)\, dx$$

$$= 2\int_{-1}^{1} e^x\, dx = 2e^x \Big|_{-1}^{1}$$

$$= 2e - 2e^{-1}$$

#29. Find the volume of the solid in the first octant enclosed by the cylinder $z = 16 - x^2$ and the plane $y = 5$



$$\therefore \text{Vol} = \int_0^4 \int_0^5 (16 - x^2)\, dy\, dx$$

$$= \int_0^4 5(16 - x^2)\, dx$$

The solid E can be represented by

$0 \le y \le 5$, $0 \le x \le 4$,

and $0 \le z \le 16 - x^2$

$$\therefore \text{Vol} = \int_0^4 \int_0^5 16 - x^2 \, dy \, dx$$

$$= \int_0^4 16y - x^2 y \Big|_0^5 \, dx$$

$$= \int_0^4 80 - 5x^2 \, dy = 80x - \frac{5x^3}{3} \Big|_0^4$$

$$= 320 - \frac{320}{3}$$