15.3 We learned how to integrate a function defined on a rectangle R

If $R = \left\{ (x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$,

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

or

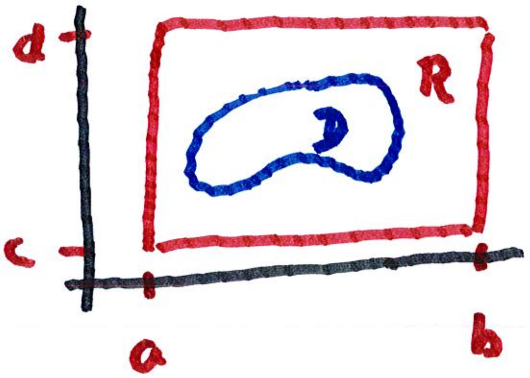
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

What if R is replaced by

a more complicated region?

We can write

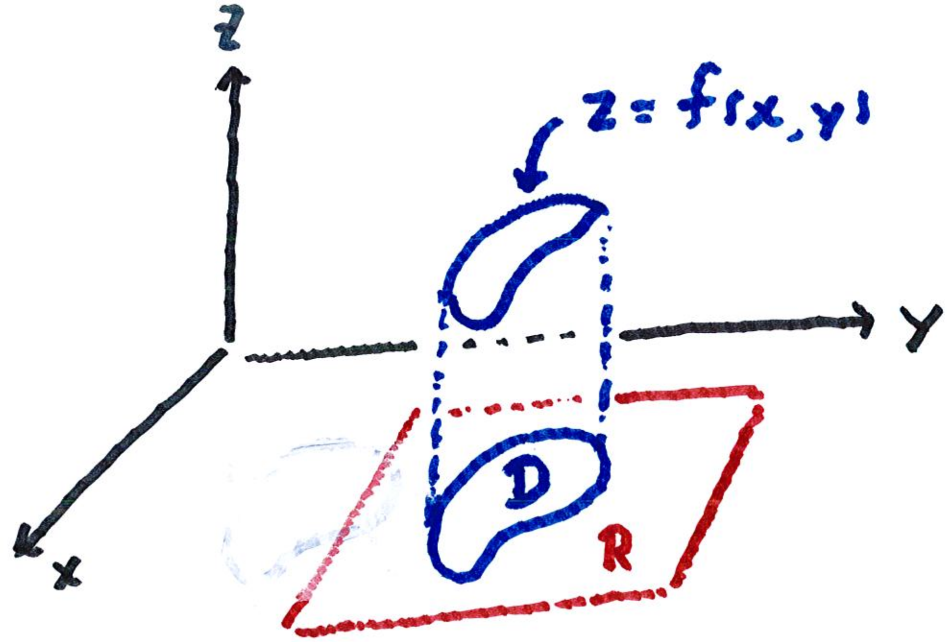
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \\ & \text{but not in } D \end{cases}$$



Geometrically

$$\iint f(x,y)\, dA = \text{volume of region}$$

above D, under $z = f(x,y)$

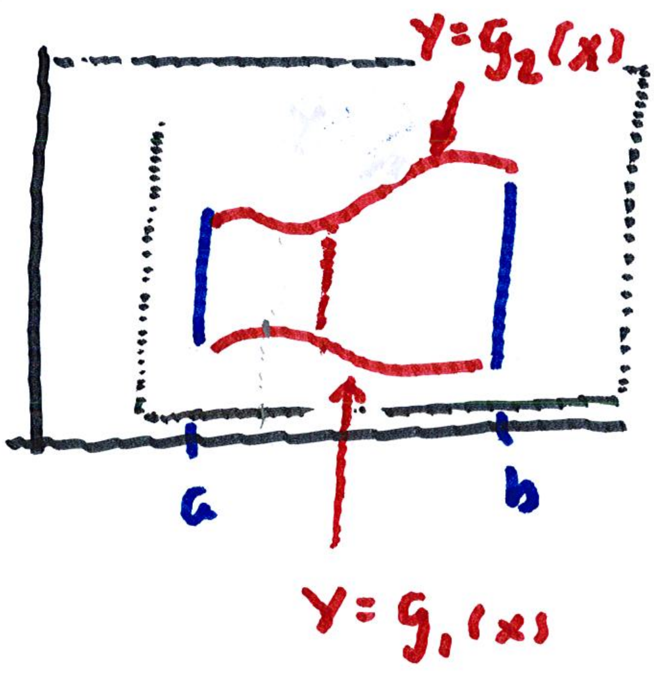


A plane region D is of type I

if it lies between the graphs

of two functions:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$



$$= \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{y\text{-integral}}\,dx$$

Ex Let $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le x \end{array} \right\}$

Compute $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2 \, dy \, dx$

$$= \int_0^1 \left. \frac{xy^3}{3} \right|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x \cdot x^3}{3} - \frac{x \cdot x^6}{3} \right) dx$$

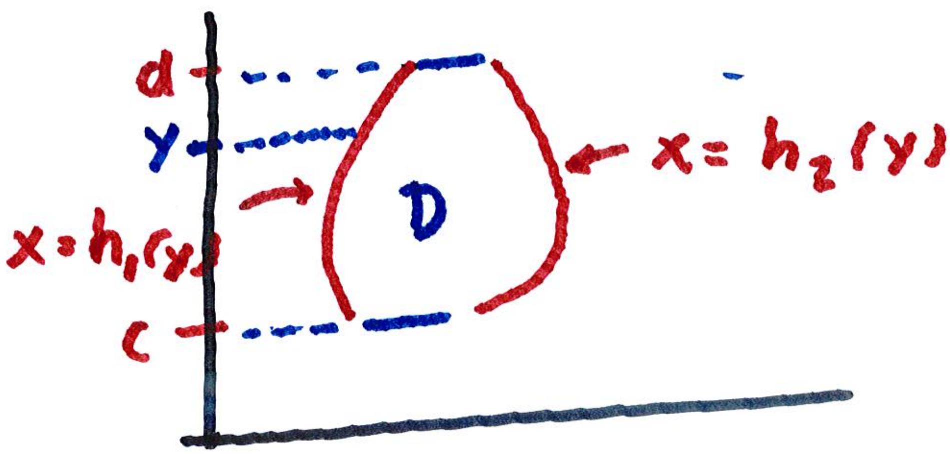
$$= \int_0^1 \frac{x^4}{3} - \frac{x^7}{3} \; dx$$

$$= \frac{x^5}{15} - \frac{x^8}{24} \Big|_0^1$$

$$= \frac{1}{15} - \frac{1}{24} = \frac{8-5}{120} = \underline{\underline{\frac{1}{40}}}$$

A region D is of type II

if $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right.$



Ex. Let D be the region between

the graphs of $x = y^2$ and $x = 2y+3$

Compute $\displaystyle\iint_D y + x \, dA$

When do $x=y^2$ and $x=2y+3$

coincide?

$y^2=x=2y+3$

$\rightarrow \quad y^2-2y-3=0$

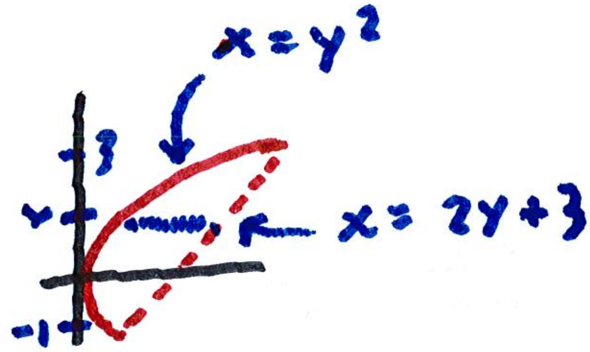
$(y-3)(y+1)=0 \quad y=-1,\ y=3$

Which curve is above?

Plug $y=0$ into both

equations $\quad 0=x \quad 2\cdot 0+3=y$

$\therefore \quad 2y+3$ is $>$ y^2



$$\rightarrow \int_{-1}^{3} \int_{y^2}^{2y+3} y + x \, dx \, dy$$

$$= \int_{-1}^{3} xy + \frac{x^2}{2} \Bigg|_{x=y^2}^{x=2y+3} dy$$

$$= \int_{-1}^{3} (2y+3)y + \frac{(2y+3)^2}{2} - y^2 y - \frac{y^4}{2} \, dy$$

$$= \int_{-1}^{3} \frac{2y^3}{3} + 3y + 2y^2 + 6y + \frac{9}{2} - y^3 - \frac{y^4}{2}$$

$$= \int_{-1}^{3} \frac{-y^4}{2} - y^3 + \frac{8y^2}{3} + 9y + \frac{9}{2} \, dy$$

$$= -\frac{y^5}{10} - \frac{y^4}{4} + \frac{8y^3}{9} + \frac{9y^2}{2} + \frac{9y}{2}\Bigg|_{-1}^{3}$$

$$= \cancel{???} \quad e^{\pi}$$

Ex. Evaluate $\displaystyle\iint_D xy \, dA$.

where

D is bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$

$y^2 = x$ $\qquad$ $x = 2y$

Usually it's better to avoid

square roots

$y^2 = x = 2y$

$\rightarrow \; y^2 - 2y = 0 \rightarrow \begin{cases} y = 0 \\ y = 2 \end{cases}$

Plug $y = 1$ $\quad y^2 = 1 \quad 2 \cdot 1 = 2$

$\therefore$ $x = 2y$ is bigger (in x-direction

$$\int_0^2 \int_{y^2}^{2y} xy \, dx \, dy$$

$$= \int_0^2 \frac{x^2 y}{2} \Big|_{y^2}^{2y} \, dy$$

$$= \int_0^2 \frac{(2y)^2 y}{2} - \frac{(y^2)^2 y}{2} \, dy$$

$$= \int_0^2 2y^3 - \frac{y^5}{2} \, dy$$

$$= \frac{y^4}{2} - \frac{y^6}{12} \Big|_0^2 = \frac{8}{3}$$

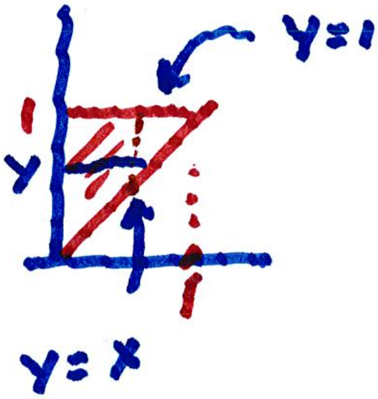
Ex. Sometimes only Type I or

Type II is possible

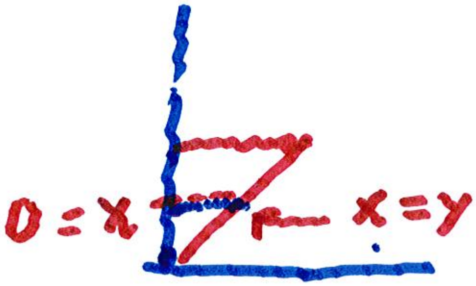
Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$

$$= \iint_D \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \int_x^1 \sin y^2\, dy\, dx$$



$= ?$



$$\iint_{D} \sin(y^2)\, dA$$

$$= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \Big[x \sin(y^2) \Big]_{x=0}^{x=y}$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= \frac{1}{2} \int_0^1 \sin(y^2)\, 2y \, dy$$

$$= -\frac{1}{2} \cos(y^2) \Big|_0^1$$

$$= -\frac{1}{2} \cos 1 + \frac{1}{2} \cos 0$$

$$= \frac{1}{2} (1 - \cos 1)$$

Properties of Double Integrals

$$\iint_D [f(x,y) + g(x,y)]\, dA$$

$$= \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

And:

$$\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$$

If $f(x,y) \ge g(x,y)$, for (x,y) in D,

then

$$\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$$

Also if $D = D_1 \cup D_2$, where

D_1 and D_2 don't intersect, then (except at boundaries)

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

$$\iint_D 1 \, dA = \text{Area of } D$$

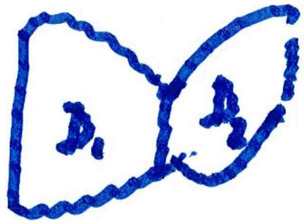
and if

$$m \le f(x,y) \le M. \quad \text{for all } (x,y) \text{ in } D.$$

then

$$m \, \text{Area}(D) \le \iint_D f(x,y) \, dA$$

$$\le M \, \text{Area}(D)$$



$D = D_1 \cup D_2$

31. Find volume of region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ in the first octant.

The region on the x-y plane

is $\left\{ (x,y) : \begin{array}{l} 0 < x < 1 \\ 0 < y < \sqrt{1-x^2} \end{array} \right\} = D^+$

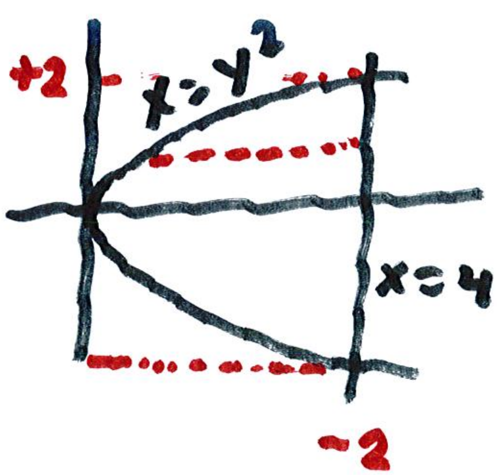
$$\text{Vol} = \int_0^1 \int_0^{\sqrt{1-x^2}} y \, dy \, dx$$

$$= \int_0^1 \frac{y^2}{2} \Bigg|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2} \, dx = \frac{x}{2} - \frac{x^3}{6} \Bigg|_0^1$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Find the volume of the region under the surface $z = 1 + x^2 + y^2$ and above the region enclosed by $x = 4$ and $x = y^2$



$y^2 = x = 4$

$y = \pm 2$