15.8 Triple Integrals in Cylindrical Coordinates

In the plane we can use polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In 3 dimensions, we can use a related system of polar coordinates

If (r, θ, z) are cylindrical coord.

then the rectangular coord. are

$$x = r\cos\theta \quad y = r\sin\theta \ , \ z = z$$

and if (x, y, z) are rectangular

coord., then the cylindrical coord are

$$r^2 = x^2 + y^2 \ , \ \tan\theta = y/x \ , \ z = z.$$

Ex. If a point P has cylindrical coord. $= (2, \frac{\pi}{6}, 3)$,

then the rectangular coord. are

$$x = 2 \cdot \cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2\sin\frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$$

$$z = 3$$

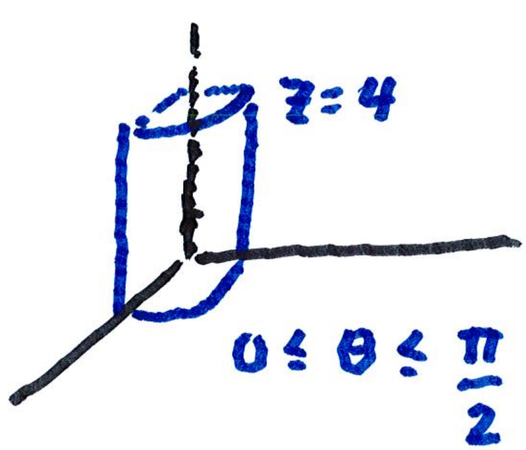
Ex. What are the cylindrical coord. of the surface

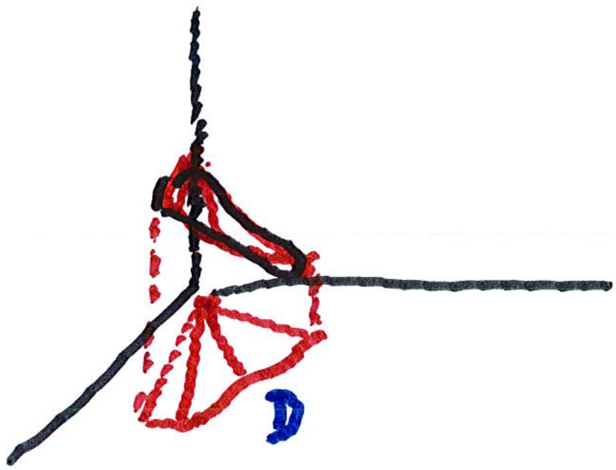
Ex: $x^2+y^2=6$, in the first octant

with $0 \le z \le 4$

$r^2=6$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le z \le 4$

$\therefore (r, \theta, z) = (\sqrt{6}$, $0 \le \theta \le \frac{\pi}{2}$. $0 \le z \le 4$





$$\iiint_E f(x,y,z)\,dV$$

$$= \iint_D \left[\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(r\cos\theta,\, r\sin\theta,\, z)\,dz \right] dA$$

$dA = r\,dr\,d\theta$

Which equals

$$\iint_D \left[\int_{u_1(r\cos\theta,\, r\sin\theta}^{u_2(r\cos\theta,\, r\sin\theta} f(r\cos\theta,\, r\sin\theta,\, z\} \right] r\,dr\,d\theta$$

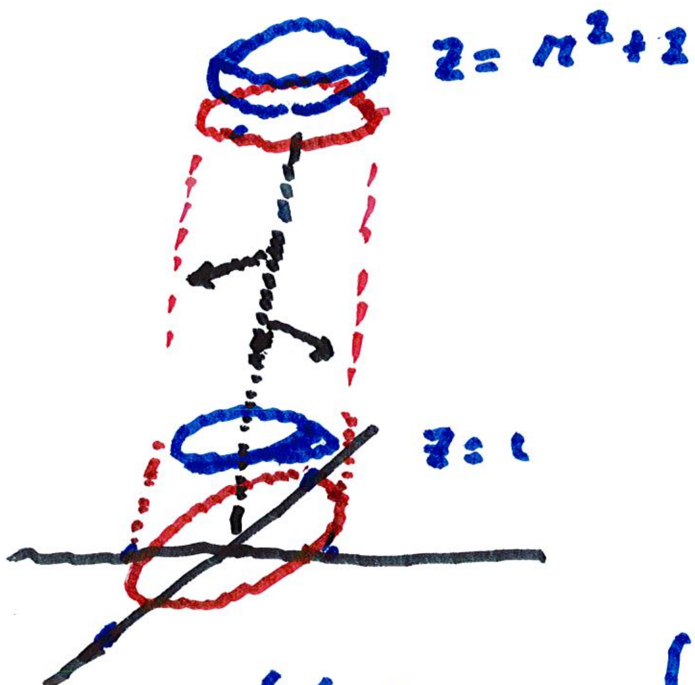
Where we used the formula

$$dA = r\,dr\,d\theta.$$

Ex. Find $\iiint_E z\,dA$,

where E is bounded by

$1 \le z \le x^2+y^2+2$, $x^2+y^2=1$



$$\iiint z\,dV = \int_0^{2\pi}\int_0^1\int_1^{r^2+2} z\,dz\,r\,dr\,d\theta$$

$dz\ r\,dr\ d\theta$

$$= 2\pi \int_0^1 \int_1^{r^2+2} z \, dz \, r \, dr$$

$$= 2\pi \int_0^1 \frac{z^2}{2} \Big|_1^{r^2+2} r \, dr$$

$$= \pi \int_0^1 \left((r^2+2)^2 - 1 \right) r \, dr$$

$$= \pi \int_0^1 r^5 + 4r^3 + 3r \quad dr$$

$$= \pi \left(\frac{r^6}{6} + r^4 + \frac{3r^2}{2} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{6} + 1 + \frac{3}{2} \right) = \underline{\underline{\frac{5\pi}{3}}}$$

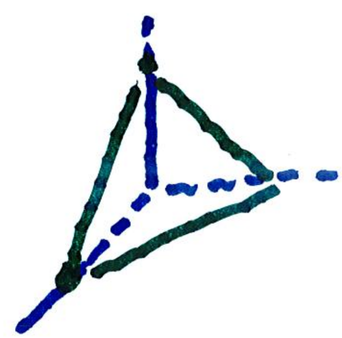
Ex. Suppose E = a solid tetrahedron bounded by

$x + y + z = 1$, and $x = 0$, $y = 0$,

and $z = 0$ (E is in first octant)

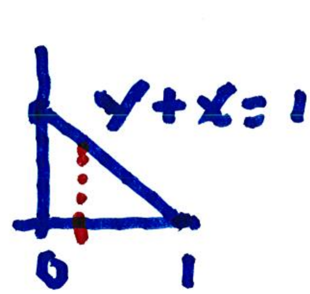
If the density of E = z,

calculate mass of E



The plane intersects $z = 0$

when $x + y = 1$



$$\text{Mass} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2} \Bigg|_{z=0}^{z=\frac{1-x-y}{2}} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (y+x-1)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \frac{(y+x-1)^3}{3} \Bigg|_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x+x-1)^3 - (0+x-1)^3\,dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\,dx$$

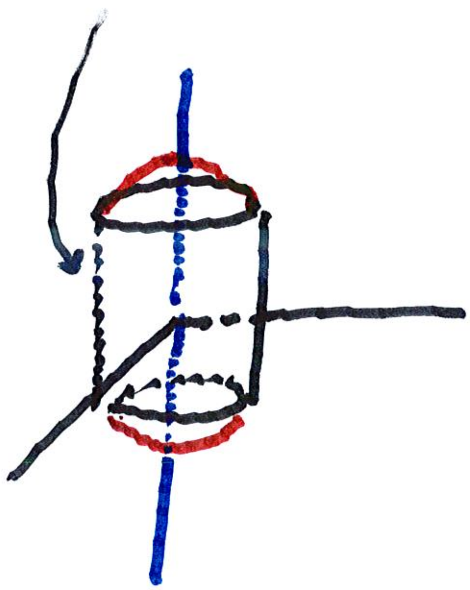
$$= -\frac{1}{6}\int_0^1 (x-1)^3\,dx$$

$$= -\frac{1}{6}\cdot\frac{(x-1)^4}{4}\Big|_0^1$$

$$= -\frac{1}{6}\cdot\left(0-\frac{1}{4}\right) = \underline{\underline{\frac{1}{24}}}$$

Ex. Find the volume of the solid that lies within both the cylinder

$x^2+y^2=1$ and the sphere $x^2+y^2+z^2=2$

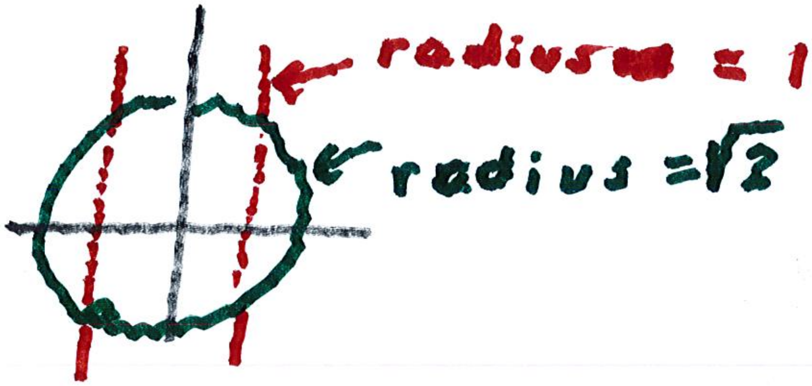


region satisfies

$0<r<1 \quad 0\le\theta\le 2\pi$

and $-\sqrt{2-x^2-y^2}<z$

and $z<\sqrt{2-x^2-y^2}$



$$\text{Vol} = \int_0^{2\pi} \int_0^{1} \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} 1 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{1} 2\sqrt{2-r^2} \quad r \, dr \, d\theta$$

$$= -\int_0^{2\pi} \int_0^{1} \sqrt{2-r^2} \, (-2r \, dr) \, d\theta$$

$$= -\int_0^{2\pi} \cdot \frac{2}{3} (2-r^2)^{3/2} \Big|_0^1$$