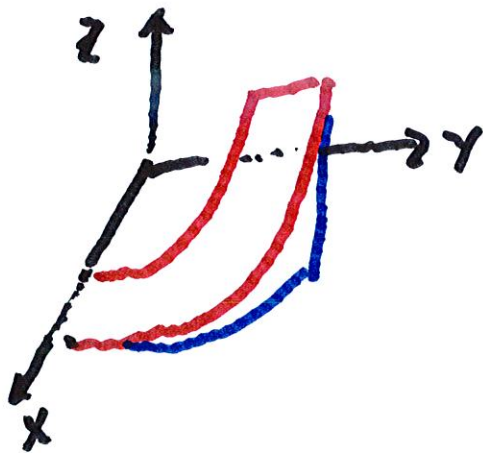


Ex. Compute $\iiint_E z \, dV$ where E

is in the first octant and lies

between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

and lies below $z = y$



$$\iiint_E z \, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \int_0^{r \sin \theta} z \, dz \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{z^2}{2} \int_0^{r \sin \theta} r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r^2 \sin^2 \theta}{2} dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{8} \sin^2 \theta \Big|_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 - \frac{1}{4}\right) \frac{\sin^2 \theta}{2} d\theta$$

$$= \frac{15}{8} \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \frac{15}{8} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{15\pi}{16} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta$$

$$= \frac{15\pi}{32} - \left(\frac{15 \sin 2\theta}{32} \Big|_0^{\frac{\pi}{2}} \right)$$

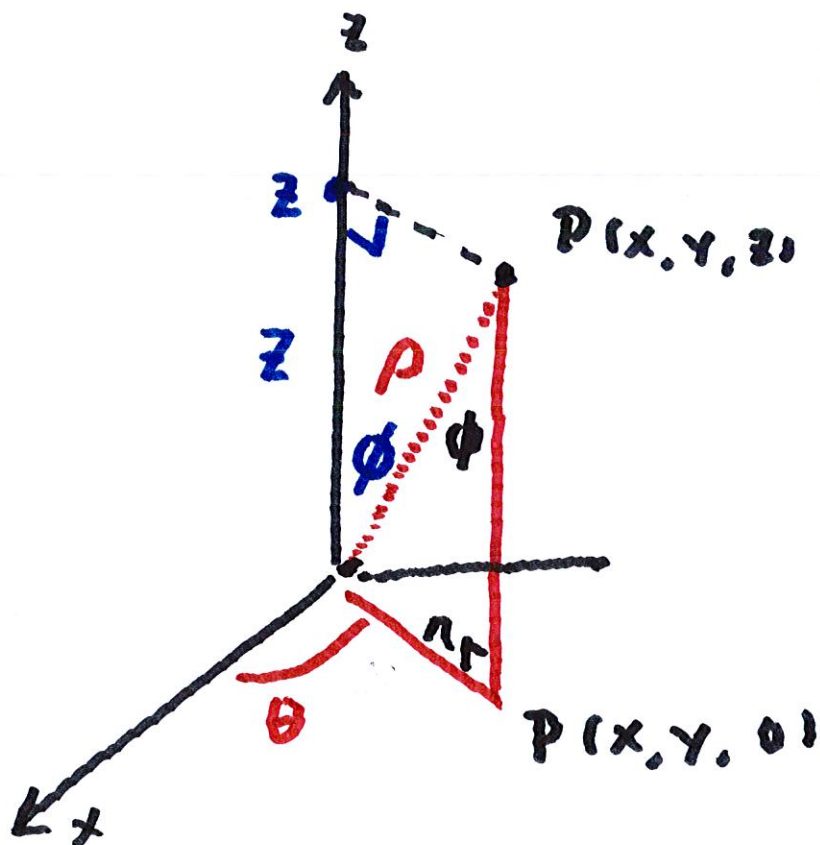
$$= \frac{15\pi}{32}$$

Ex. Let E be the region in the first octant bounded by $x^2 + z^2 = 4$ and bounded between $y = 4$ and $y = x$.

Spherical Coord.

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ϕ : angle between
P and positive
z-axis



$$\frac{z}{\rho} = \cos \phi$$

$$z = \rho \cos \phi$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$\frac{r}{\rho} = \sin \phi \rightarrow r = \rho \sin \phi$$

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta$$

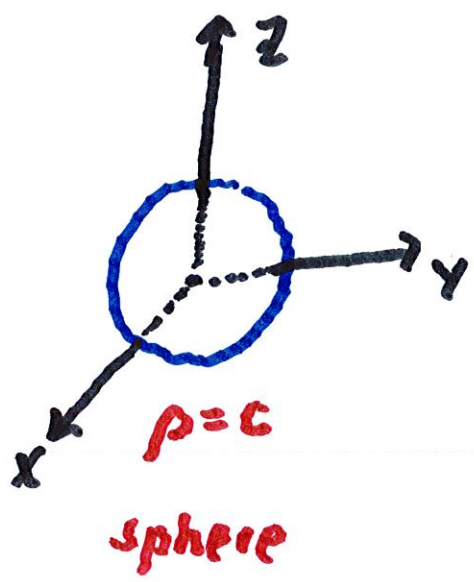
Spherical Coord (ρ, θ, ϕ)

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \text{or } \rho^2 = x^2 + y^2 + z^2$$

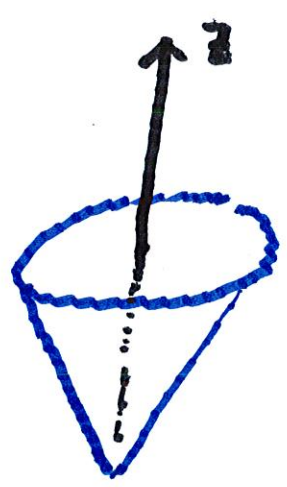
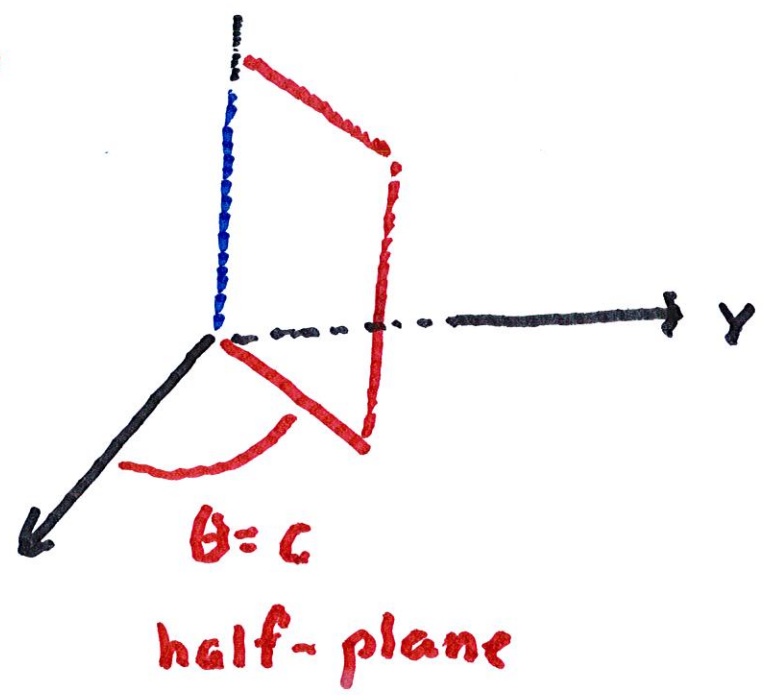
If we project (x, y, z) down to the xy -plane, then

$\theta =$ usual coordinate of $(x, y, 0)$

$$x = \rho \cos \theta, \quad y = \rho \sin \theta$$

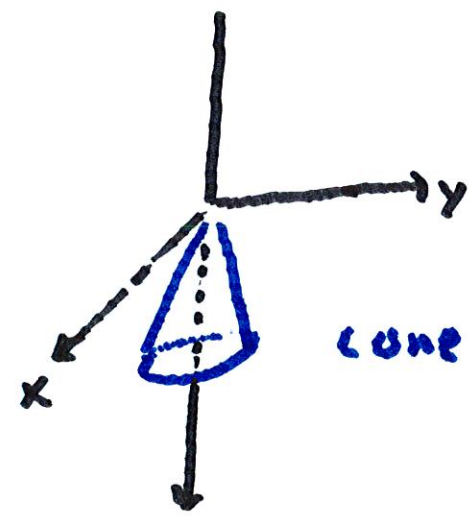


(ρ, θ, ϕ) are spherical coord:
coord:

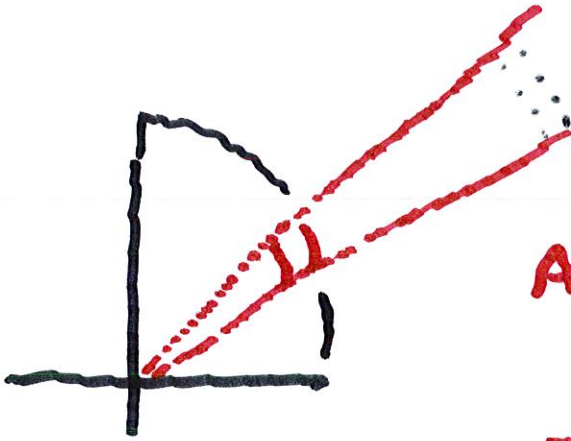


$\phi = c < \frac{\pi}{2}$ cone

$\frac{\pi}{2} < \phi < \pi$



In polar coord



Area of $\Delta r \Delta \theta$

$$= r \Delta r \Delta \theta$$

In spherical coord.

Eucl. Volume of $(\Delta r \Delta \theta \Delta \phi)$ -box

$$= (r^2 \sin \phi) (\Delta r \Delta \theta \Delta \phi)$$

∴ When converting from

(x, y, z) integral to a spherical,

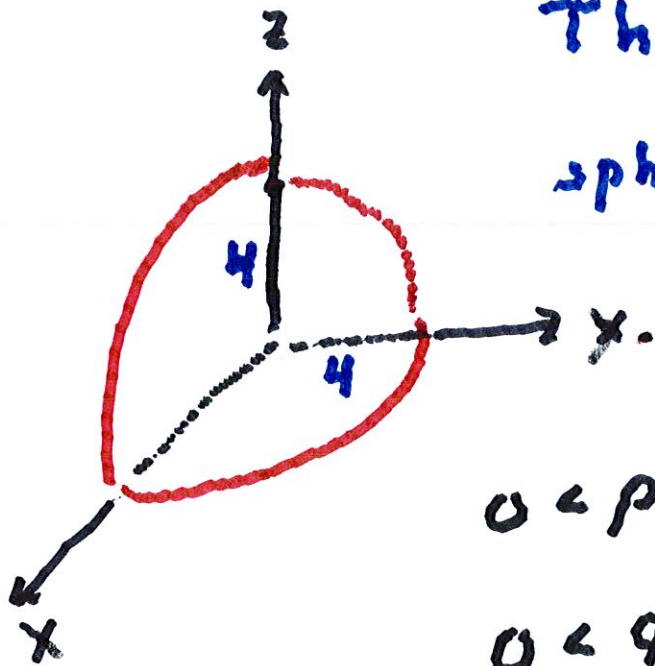
You must include $\rho^2 \sin \phi$.

Ex. Let $E =$ region of ball

(about $(0, 0, 0)$) of radius 4

in the first octant.

Compute $\iiint_E z \, dV$.



This is a box in
spherical coord.

$$0 < \rho < 4$$

$$0 < \phi < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{2}$$

$$\int \int \int_E z \, dV = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho \cos \phi \, \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \phi \cos \phi \frac{\rho^4}{4} \Big|_0^4 d\phi d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \phi \cos \phi d\phi d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2 \phi \Big|_0^{\frac{\pi}{2}} d\theta$$

$$= 32 \int_0^{\frac{\pi}{2}} d\theta = \underline{\underline{16\pi}}$$

Find the center of mass of

$$E = \left\{ (x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4, \right. \\ \left. \text{where } z \geq 0 \right\}$$

Assume density = 1

$$1 \leq \rho^2 \leq 4 \rightarrow 1 \leq \rho \leq 2$$

$$z \geq 0 \rightarrow 0 \leq \phi \leq \frac{\pi}{2}$$

$$\text{mass} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



$$\iiint_E = 2\pi \int_0^{\frac{\pi}{2}} \int_1^2 \rho^2 \sin \phi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin \phi \left. \frac{\rho^3}{3} \right|_1^2$$

$$= \frac{2\pi}{3} \cdot (8-1) \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi$$

$$= -\frac{14\pi}{3} \int_0^{\frac{\pi}{2}} -\sin \phi \, d\phi$$

$$= -\frac{14\pi}{3} \cos \phi \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{14\pi}{3} (0 - 1) = \frac{14\pi}{3}.$$

$$M_{xy} = \iiint_E z \, dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cos \phi \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\rho^4}{4} \sin \phi \Big|_1^2 d\phi$$

$$= \frac{\pi}{2} (16-1) \int_0^{\frac{\pi}{2}} \sin \phi d\phi$$

$$= \frac{15\pi}{2} (-\cos \phi) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{15\pi}{2} (0 - (-1)) = \frac{15\pi}{2}$$

$$\bar{z} = \frac{M_{xy}}{\frac{14\pi}{3}} = \frac{\frac{15\pi}{2}}{\frac{14\pi}{3}}$$

$$= \frac{45}{28}$$

$$\bar{y} = 0 \text{ and } \bar{x} = 0$$

by symmetry

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{45}{28} \right)$$

Ex. Find the volume of the

region below the sphere $x^2 + y^2 + z^2 = 1$

and above the cone $z = \sqrt{x^2 + y^2}$

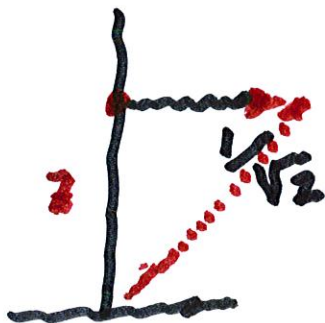
Set $r = \sqrt{x^2 + y^2}$



$$z^2 = 1 - r^2$$

and $z = r \rightarrow z^2 = r^2$

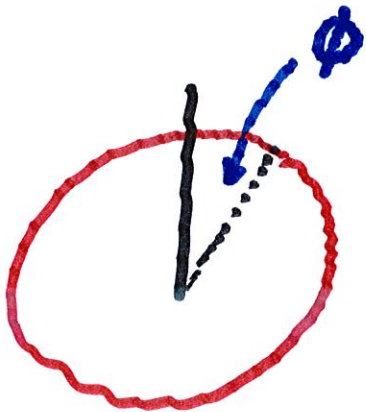
$$\therefore 1 - r^2 = r^2 \rightarrow 2r^2 = 1 \rightarrow r = \frac{1}{\sqrt{2}}$$



$$z = r \quad r = \frac{1}{\sqrt{2}}$$

$\therefore \tan \phi = \frac{1/\sqrt{2}}{1/\sqrt{2}}$

$$\Rightarrow \phi = \frac{\pi}{4}$$



$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{\rho^3}{3} \Big|_0^1 \int_0^{\pi/4} \sin \phi \, d\phi \quad d\rho \, d\phi \, d\theta$$

$$= \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$$

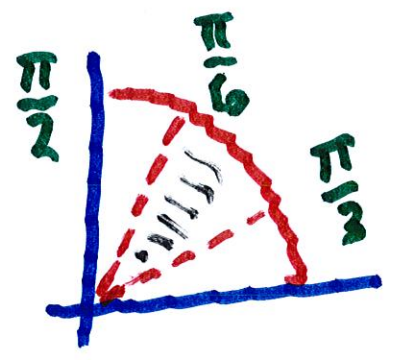
$$= \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$$

In other words

$$dV = \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

Ex. Express the volume of the ~~part of~~ the part of the ball $\rho \leq a$ that lies between

the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$



$$Vol = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi \int_0^a \rho^2 \, d\rho \, d\phi$$

$$= 2\pi \cdot \frac{a^3}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi \, d\phi$$

$$= \frac{2\pi a^3}{3} (-\cos \phi) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{2\pi a^3}{3} \left(-\frac{1}{2} - \left(-\frac{\sqrt{3}}{2} \right) \right)$$

$$= \frac{\pi a^3}{3} (\sqrt{3} - 1)$$

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Ex Find the volume of

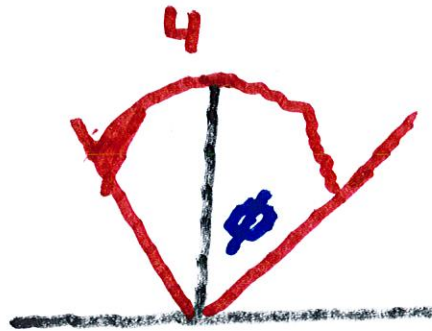
the solid that lies ~~between~~

the within the sphere

$x^2 + y^2 + z^2 = 4$, above the

xy -plane and below the cone

$$z = \sqrt{x^2 + y^2}$$



$$\frac{\pi}{4} < \phi < \frac{\pi}{2}$$

$$\text{Vol} = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{\rho^3}{3} \Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \phi \, d\phi$$

$$\rho = 2$$

$$= \frac{2\pi \rho^3}{3} \left(-\cos \phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right)$$

$$= \frac{2\pi \rho^3}{3} \left(0 + \frac{1}{\sqrt{2}} \right)$$

$$= 8\pi\sqrt{2}$$