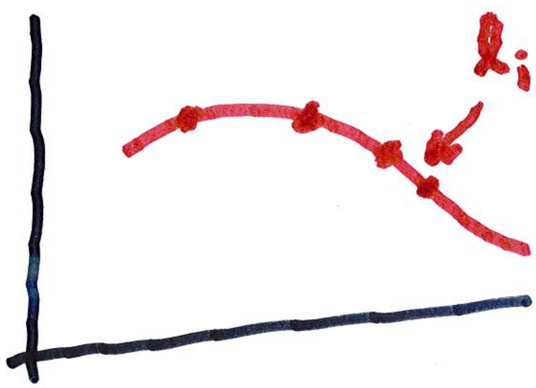
16.2 Line Integrals cont'd.

We learned that there are two kinds of line integrals

$$\int_C f(x,y) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$



ℓ_i = length of i-th segment

This is an integral of a function

Compute $\int_C x^3 y \, ds$,

where C is parameterized by

$\vec{r}(t) = \langle t^2, -3t \rangle$, $0 \le t \le 2$.

Note $\vec{r}\,'(t) = \langle 2t, -3 \rangle$

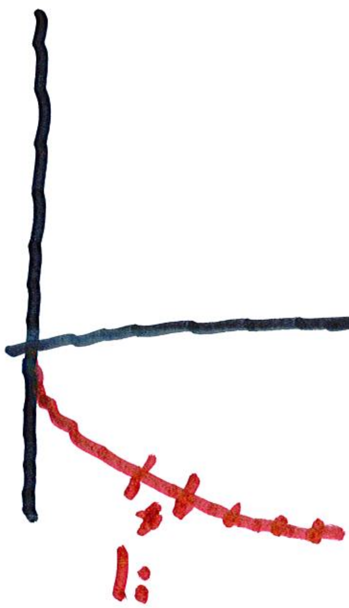
$\therefore |\vec{r}\,'(t)| = \sqrt{4t^2 + (+9)}$

$= \sqrt{4t^2 + 9}$

$$\int_C x^3 y \, ds = \int_0^2 (t^2)^3 (-3t) \sqrt{4t^2 + 9} \, dt$$

Since $\sqrt{4t^2+9}$ = speed of particle.

$\sqrt{4t^2+9}\,\Delta t$ = small amount of distance, when Δt has a small change



Ex. Compute

$$\int_C (3x - y + z)\, ds \quad \text{if}$$

$$C = \{ (\cos t, \ \sin t, \ 3t) ; \ 0 \le t \le \pi \}$$

$$\vec{r}'(t) = (-\sin t, \ \cos t, \ 3)$$

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos)^2 + 3^2}$$

$$= \sqrt{10}$$

$$\therefore \int_C = (3\cos t - \sin t + 3t)\sqrt{10}\, dt$$

$$= \left(3\sin t + \cos t + \frac{3t^2}{2} \Big|_0^{\pi} \right)\sqrt{10}$$

$$= \left[\left(-1 + \frac{3\pi^2}{2} \right) - 1 \right] \sqrt{10}$$

Another kind of line integral

is $\int_C f(x,y)\,dx$ or $\int_C g(x,y)\,dy$

Note that if $x = x(t)$, then

$$dx = x'(t)\,dt,$$

and similarly, if $y = y(t)$,

then $dy = y'(t)\,dt$

Suppose that $C = \{(t^2, 3t) : 0 \le t \le 1\}$

Compute $\displaystyle\int_C xy^2\,dx$.

$x = t^2$, and $y = 3t$, and $x'(t) = 2t$

$$\therefore \int_C xy^2\,dx = \int_0^1 t^2 (3t)^2 \cdot 2t\,dt$$

$$= \int_0^1 18\,t^5\,dt$$

$$= \frac{18}{6} = 3$$

We usually combine the integrals:

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

$$= \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Ex. Evaluate $\int_C y^3\,dx - x^2\,dy$,

where C is parameterized by

$(\cos t, \sin t)$, for $0 \leq t \leq \pi$

$\uparrow$ $x(t)$ $\quad$ $\uparrow$ $y(t)$

$x' = -\sin t \qquad y' = \cos t$

$$\int_0^{\pi} (\sin t)^3 \cos t - (\cos^2 t) \cos t \, dt$$

$$\frac{\sin^4 t}{4} \Big|_0^{\pi} \qquad - (1 - \sin^2 t) \cos t \, dt$$

$$- \sin t + \frac{\sin^3 t}{3} \, dt$$

$\therefore$

$$\int_C = \qquad 0 - \sin t + \frac{\sin^3 t}{3} = 0$$

For a curve in space, suppose that $x = x(t)$, $y = y(t)$, $z = z(t)$, for $a \le t \le b$.

$$\int_C f(x, y, z)\, ds$$

$$= \int_a^b f(x(t), y(t), z(t))\, dt \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

dt

One can also define expressions such as

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z),$$

where one multiplies each term by $x'(t)$, $y'(t)$, and $z'(t)$.

Now we define line integrals of a vector field.

Recall that if $\vec{F}$ is a vector, and if $\vec{v}$ is a unit vector, then $\vec{F} \cdot \vec{v}$ is the component of $\vec{F}$ in the $\vec{v}$-direction. For example, if $\vec{F} \in \mathbb{R}^3$, then

$\vec{F} \cdot \langle 0, 1, 0 \rangle = F_2$, and

$\vec{F} \cdot \langle 0, 0, 1 \rangle = F_3$, and

$$\frac{1}{\sqrt{2}} F_1 + \frac{1}{\sqrt{2}} F_2 \text{ is}$$

the component of $\vec{F}$ in the $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$ direction.

Now let C be a curve in

$\mathbb{R}^2$ (or $\mathbb{R}^3$) is defined by

$\vec{r}(t) = \langle x(t), y(t) \rangle$.

The velocity at time t is

$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$.

The work done during a Δt_i

time interval is

$$\vec{F}(x_i, y_i) \cdot \underbrace{\frac{\vec{r}\,'(t_i)}{|\vec{r}\,'(t_i)|}}_{= \vec{T}(t_i)} \underbrace{|\vec{r}\,'(t_i)|\,\Delta t_i}_{s_i}.$$

Letting $n \to \infty$, and summing up over all time intervals,

we get

$$W = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

Since $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$.

we get

$$W = \int_a^b P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)$$

This is the line integral of

the force $P(x,y)\vec{i} + Q(x,y)\vec{j}$

along a curve $C = (x(t), y(t))$

Ex. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where

$\vec{F} = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$, and

C is the twisted cubic

$x = t,\quad y = t^2,\quad z = t^3,\quad 0 \le t \le 1$

$$\int_0^1 t\cdot t^2\cdot 1 + t^5\cdot 2t + t^4\cdot 3t^2 \; dt$$

$$= \frac{1}{4} + \frac{2}{7} + \qquad + \frac{3}{7} = \frac{1}{4} + \frac{5}{7}$$

$$= \frac{27}{28}$$

Ex. Compute $\int_C z^2\,dx + x^2\,dy + y^2\,dz$,

where C = line segment from

$(1,0,0)$ to $(4,1,2)$

$\vec{r}(t) = \langle 1,0,0\rangle + t\langle 3,1,2\rangle$.

$\therefore\ x(t) = 3t-1$

$y(t) = t$

$z(t) = 2t$

Then the integral is

$$\int_0^1 (4t^2)\cdot 3 + (3t+1)^2\cdot 1 + t^2\cdot 2t\ dt$$

$$= \int_0^1 12t^2 + 9t^2 + 6t + 1 + 2t^3 \, dt$$

$$= \frac{21}{3} + 3 + 1 + \frac{1}{2} = \frac{23}{2}$$

Note: When C is parameterized with the opposite orientation, the line integral $\int_{-C} \vec{F} \cdot d\vec{r}$, the sign changes by (-1)

$$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$$

Ex. Compute $\displaystyle\int_C x\,dx + y\,dy + xy\,dz$,

if C is defined by

$\vec{r}(t) = (\cos t)\vec{i} + \sin t\,\vec{j} + t\vec{k}$

for $0 \le t \le 1$.

$x = \cos t \qquad y = \sin t \qquad z = t$

$x' = -\sin t \qquad y' = \cos t \qquad z' = 1$

$$\int_C = \int_0^1 \cos t(-\sin t) + \sin t(\cos t) + \frac{t}{1}$$

$$= t\Big|_0^1 = 1$$