16.3 Fundamental Thm. for Line Integrals.

Given a function $F(x)$, $a \le x \le b$, the Fundamental Thm of Calculus says:

$$\int_a^b F'(x) = F(b) - F(a).$$

For vector fields, there is

a similar statement,

Given a function $f(x,y)$

$\left(\text{or also } f(x,y,z) \right)$ we can define

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$\left(\text{or also } \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \right)$$

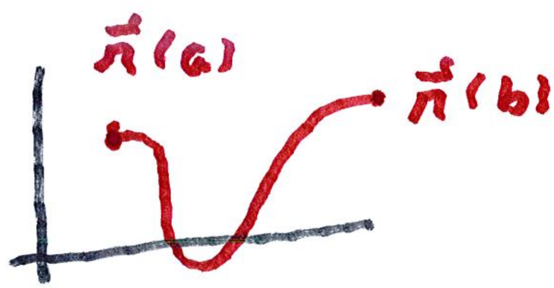
called the gradient of f.

Thm. Suppose $f(x,y)$ is C^1

near the curve $\vec{r}(t)$, for $a \le t \le b$.

Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(b)).$$



If $\vec{r}(a) = \langle x_1, y_1 \rangle$

and $\vec{r}(b) = \langle x_2, y_2 \rangle$

In 3 variables if

$\vec{r}(a) = (x_1, y_1, z_1)$

and $\vec{r}(b) = (x_2, y_2, z_2)$, then

$$\int_C \nabla f \cdot d\vec{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Pf. $\displaystyle\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f_{(r(t))} \cdot \vec{r}\,'(t)$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\, dt$$

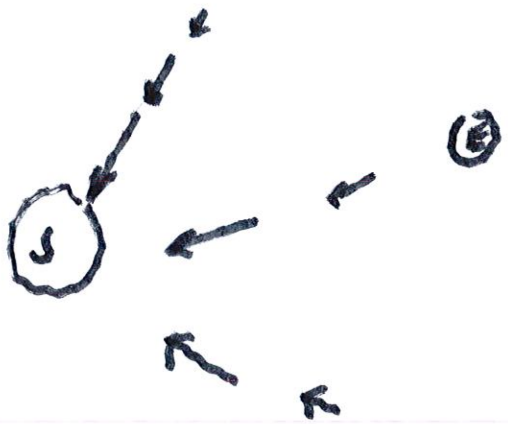
$$= f(x(t), y(t), z(t)) \Big|_a^b$$

$$= f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Ex. Gravitational field

The gravitational field is

$$\vec{F}_g(\vec{x}) = -\frac{mMG\vec{x}}{|\vec{x}|^3}$$



M: mass of sun

m: mass of earth

Set $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$

Then $\nabla f = \overline{F}_g$

If an object moves from

$(3, 4, 12)$ to $(2, 2, 0)$

then the work done is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(2,2,0) - f(3,4,12)$$

$$= \frac{mMG}{\sqrt{2^2+2^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}}$$

$$= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

Thus, if $\vec{F} = \nabla f$, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

is **independent of path.**

A curve C is closed if

$$\vec{r}(b) = \vec{r}(a)$$

If $\int \vec{F} \cdot d\vec{r}$ is independent of path, then if C_1 and C_2 are paths from A to B, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} .$$

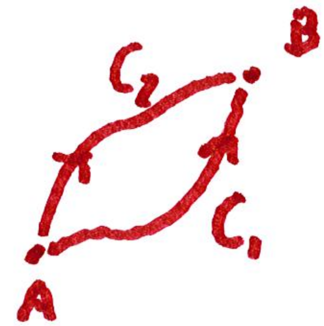
We can define a closed path by $C = C_1 + (-C_2)$. But

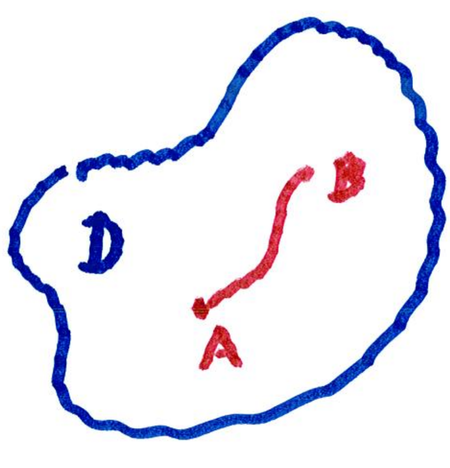
$$\int_{-C_2} \vec{F} \cdot d\vec{r} = - \int_{C_2} \vec{F} \cdot d\vec{r}.$$

$$\therefore \quad \int_{C} \vec{F} \cdot d\vec{r} =$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} \; - \int_{C_2} \vec{F} \cdot d\vec{r} \; = 0, \text{ i.e}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot dr$$





If $\int_C \vec{F} \cdot dr$ is

independent of path, then

there is a function $f(x,y)$ in D

so that

$$\frac{\partial f}{\partial x}(x,y) = P(x,y) \quad \text{and}$$

$$\frac{\partial f}{\partial y}(x,y) = Q(x,y) .$$

In fact, let C_1 be a path from

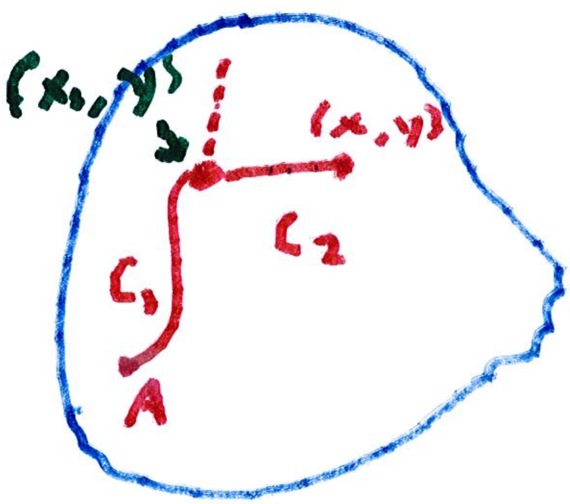
A to (x_1, y) and let C_2 be a

path (segment) from

(x_1, y) to (x,y).

$$\therefore f(x,y) = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{x_1}^{x} \vec{F}\cdot d\vec{r}$$

Hence $\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt$



$= P(x,y)$ for all x on the segment.

Now let C_2 = a straight path. A similar argument shows that $\frac{\partial f}{\partial y}(x,y) = Q(x,y)$

We say that $\vec{F} = P\vec{i} + Q\vec{j}$

is a conservative vector

field if there is a function f

so that $\nabla f = \vec{F}$, i.e.,

so that $\frac{\partial f}{\partial x} = P(x,y)$ and

$$\frac{\partial f}{\partial y} = Q(x,y).$$

How do we know if $\vec{F} = \nabla f$

for some f ?

If $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ If $\nabla f = P(x,y)\vec{i} + Q(x,y)\vec{j}$,

then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

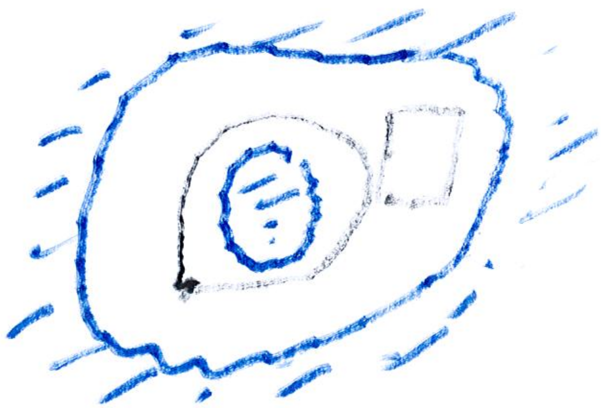
But what if the region

looks like this?

We say a region is ~~simply-connected~~

simply-connected if it has no

holes.



If D is not simply-connected

it may be that even though

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

, There is no

f so $\nabla f = P\mathbf{i} + Q\mathbf{j}$

When the ~~vector field~~ domain D is elementary, such as a disk or a rectangle, then whenever P, and Q satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$,

then the above method works and one can find f so

$$\frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q.$$

Ex. Find $f(x,y)$ so that

(2) $$\frac{\partial f}{\partial x} = 2xy - 3y \quad \text{and}$$

$$\frac{\partial f}{\partial y} = x^2 - 3x + 3y^2$$

Find f so (2) holds

Just integrate in x:

$$f(x,y) = x^2 y - 3xy + h(y).$$

Now differentiate in y

$$\frac{\partial P}{\partial y} = 2x - 3 \qquad \frac{\partial Q}{\partial x} = 2x - 3$$