13.1 Vector Functions

A vector function assigns a vector

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

to each t in the domain of $\vec{r}$. Usually the domain is the set of t such that $f(t)$, $g(t)$, and $h(t)$ are all defined.

Ex. Define

$$\vec{r}(t) = \left\langle \sqrt{t+1},\ \ln(2-t),\ \frac{1}{t^2} \right\rangle$$

Find the domain of $\vec{r}$.

t must satisfy

$t+1 \geq 0$, $2-t > 0$, and $t \neq 0$

or $t \geq -1$, $t < 2$, and $t \neq 0$

$\therefore \text{Dom}(\vec{r}) = \{-1 \leq t < 2,\ t \neq 0\}$

We define

$$\lim_{t \to a} \langle f(t), g(t), h(t) \rangle$$

$$= \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle,$$

(provided that these

3 limits exist).

Ex. Compute

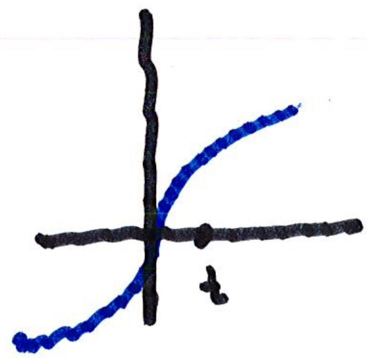
$$\lim_{t \to 0^+} \left\langle t \ln t, \sqrt[3]{t}, 2t+3 \right\rangle$$

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}}$$

$$\underset{\text{L'Hop}}{=} \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \to 0^+} (-t)$$

$$= \underline{\underline{0}}$$

$$\lim_{t \to 0^+} \sqrt[3]{t} = 0$$



$$\lim_{t \to 0^+} (2t+3) = 2 \cdot 0 + 3 = 3$$

$\therefore$ limit $= (0, 0, 3 = \underline{\underline{3}}$

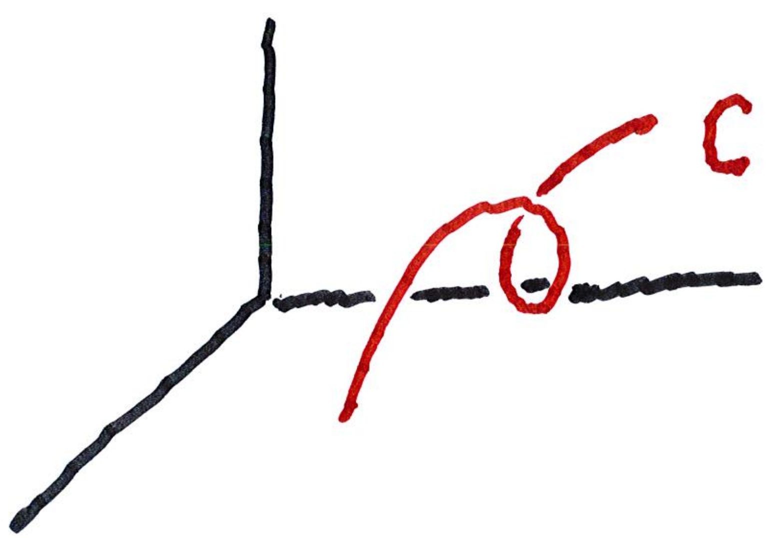
A function $\vec{r}(t)$ is continuous at a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

Suppose that $f(t)$, $g(t)$ and $h(t)$ are all continuous ~~at~~ at all t in an interval I.

We say that a set C of points (x, y, z) ~~AA~~ is a space curve if for each $(x, y, z) \in C$ there is a number t such that

$x = f(t)$, $y = g(t)$, and $z = h(t)$.



A line segment S is a space curve. Let $r_0 = (x_0, y_0, z_0)$ and $r_1 = (x_1, y_1, z_1)$.

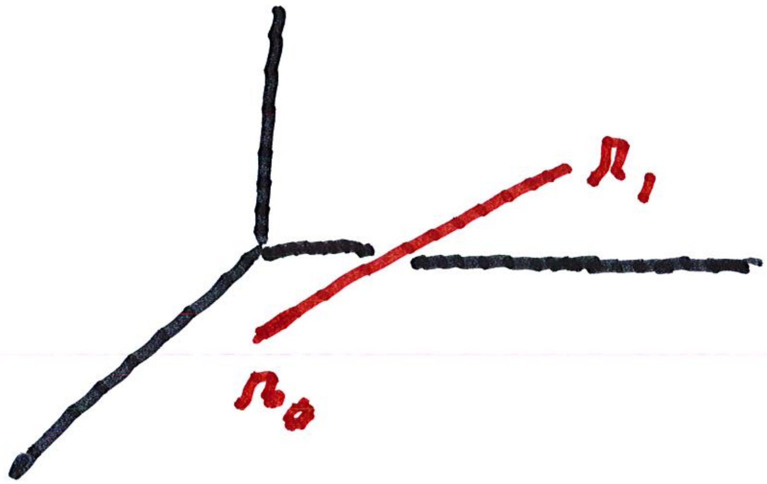
Define $\vec{v} = \overrightarrow{r_0 r_1}$

Let $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, for $0 \le t \le 1$.

$$\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

$\vec{r}(0) = r_0$ and $\vec{r}(1) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)$

$= \vec{r}_1$



$$\therefore \vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

Ex. Paramaterize the segment between $(2, 3, 1)$ and $(-1, 9, 10)$

$\vec{v} = \langle r_1 - r_0 \rangle$

$= \langle -1, 9, 10 \rangle - \langle 2, 3, 1 \rangle$

$= \langle -3, 6, 9 \rangle$

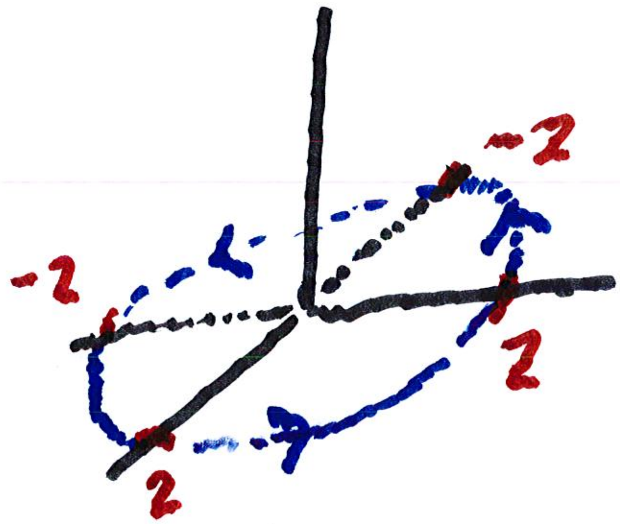
$\therefore \vec{r}(t) = \langle 2, 3, 1 \rangle + t\langle -3, 6, 9 \rangle$

Ex. Sketch the curve

$x = 2\cos t \quad y = 2\sin t, \quad z = t$

First suppose that

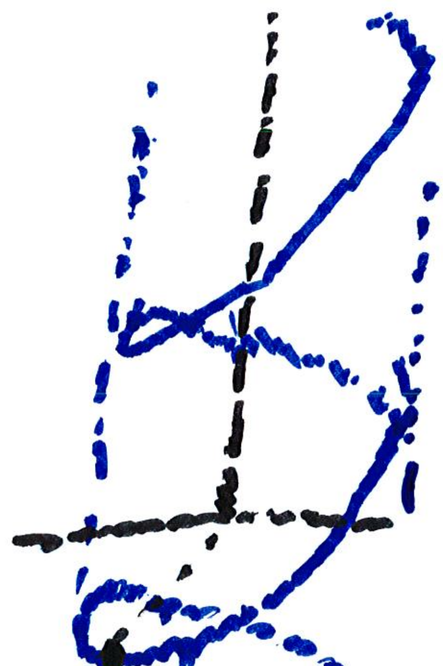
$z(t) = 0$



$(2\cos t, 2\sin t)$ goes in

a circle of radius 2.

If $z(t) = t$



We get a helix that winds up around the z-axis.

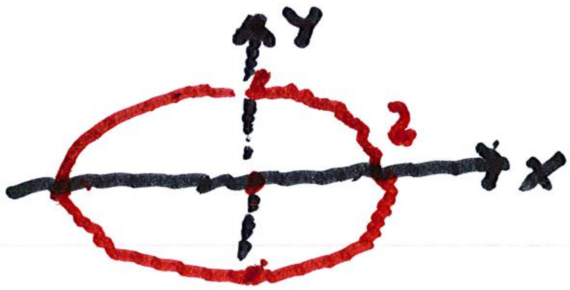
Ex. Find a parametric curve described by

$x^2 + 4y^2 = 4$ and $z = x^2$

Divide by 4

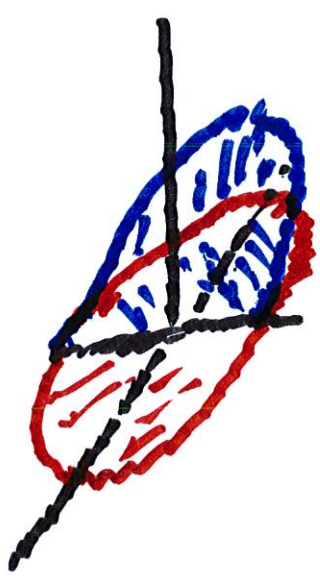
$\frac{x^2}{4} + y^2 = 1$ and $z = x^2$

Left Curve



Set $x = 2\cos t$ $\rightarrow$ $z = x^2$

and $y = \sin t$ $z = 4\cos^2 t$

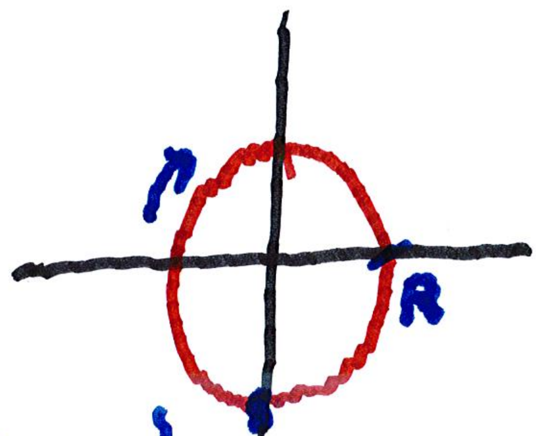


Cycloid (Rolling Wheel)

Imagine a point on a

tire of radius R

at $\theta = \frac{3\pi}{2}$



$$x = R\cos\left(\frac{3\pi}{2} - t\right)$$

$$y = R\sin\left(\frac{3\pi}{2} - t\right)$$

Use formulas for

$\cos(A-B)$ and $\sin(A-B)$:

we get $x = -R\sin t$

and $y = -R\cos t$

If wheel rests on x-axis

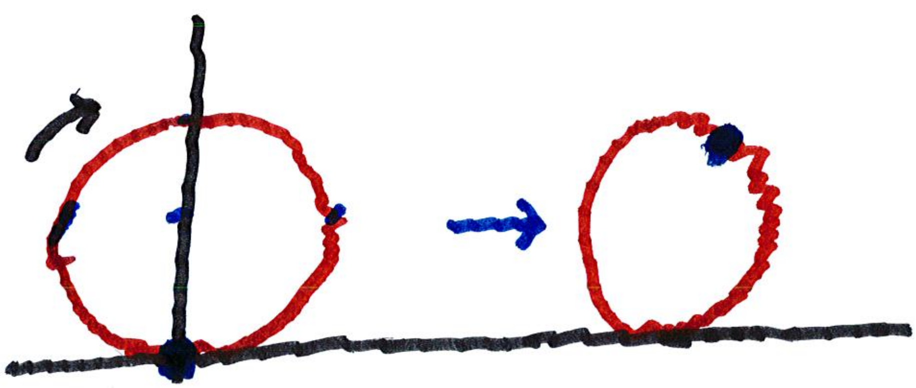
$x = -R\sin t$

$y = R - R\cos t$

After t seconds, the wheel moves Rt to the right:

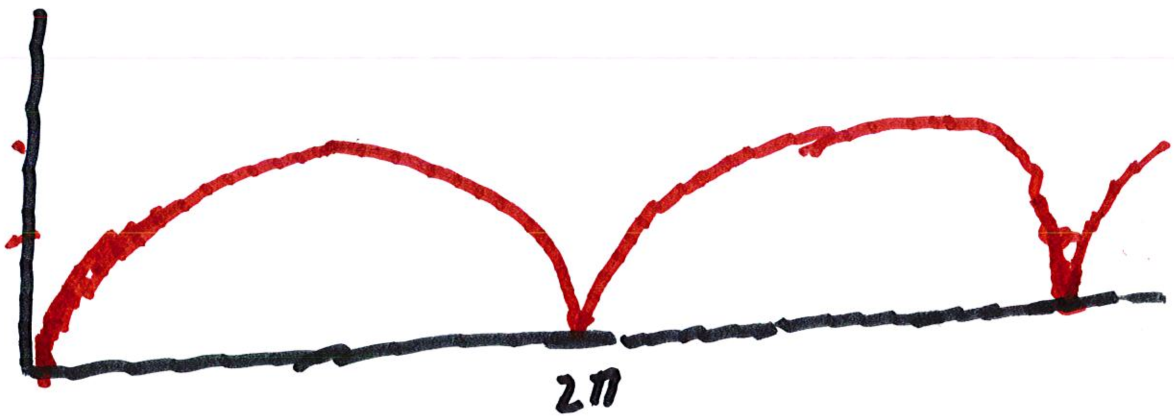
$$x = Rt - R\sin t$$

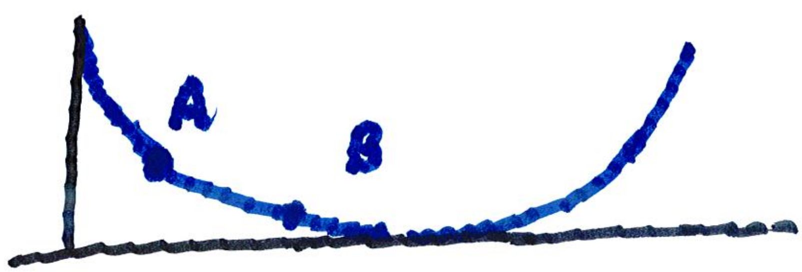
$$y = R - R\cos t$$



Locatson of Blue Spot

(Cycloid)





Among all curves joining

A and B, the cycloid takes

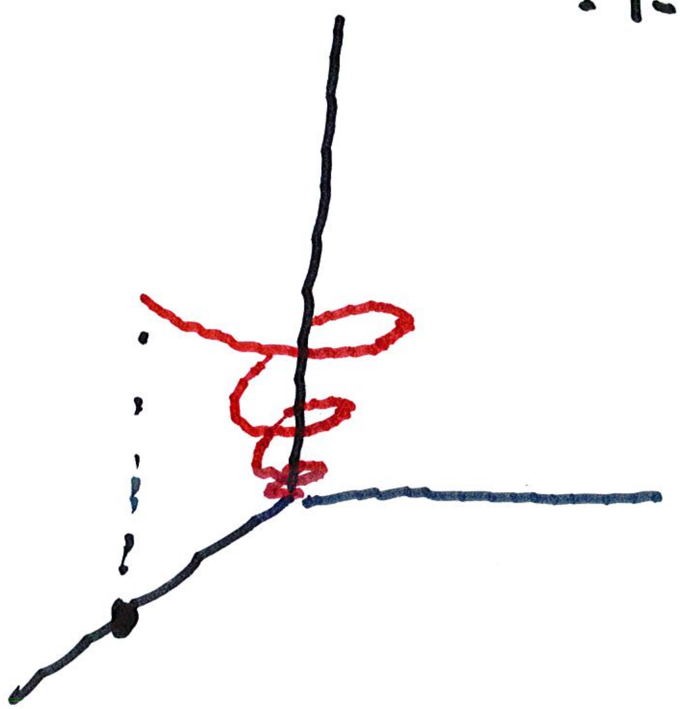
the least time. (Bernoulli)

Ex. Sketch $x = e^{-t} \cos 10t$

$y = e^{-t} \sin 10t$

$z = \cancel{t}$, $t > 0$

$= 1 - t$



Ex. Two particles have trajectories given by

$$\vec{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle$$

$$\vec{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

Looking at x and z are equal in $\vec{r}_1(t)$, it must be that

$4t - 3 = 5t - 6$

$\therefore \quad t = 3$

$\vec{r}_1(3) = \langle 9, 9, 9 \rangle$

Also

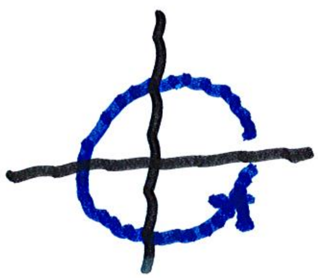
$\vec{r}_2(3) = \langle 9, 9, 9 \rangle$

They do collide !!!

Ex. Find a vector function

that represents:

$x^2 + y^2 = 4, \quad z = xy$



$x = 2\cos t \;, \; y = 2\sin t$

$\therefore \; z(t) = 4\cos t \sin t$