13.4 Velocity and Acceleration

Suppose the position of a particle at time t is $\vec{r}(t)$.

After h seconds, the distance it travels is $|\vec{r}(t+h) - \vec{r}(t)|$

$\therefore$ The speed is roughly

$$\frac{|\vec{r}(t+h) - \vec{r}(t)|}{h} \approx \left| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right|$$

As $h \to 0$, the limit is the speed

$\left| \vec{r}\,'(t) \right|$. If we focus on

$\vec{r}\,'(t)$, we get a vector

that gives the direction of

motion and its speed.

Thus, $\vec{r}\,'(t)$ is the velocity of

the particle.

Similarly, $\vec{v}'(t) = \vec{a}(t)$ is

the acceleration of the particle.

We can rewrite this as

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

Also $|\vec{v}(t)| = |\vec{r}'(t)|$ is

the speed.

Ex. Suppose $\vec{r}(t) = (2t+1, 3t^2)$

is the position at time t.

Compute it's velocity, speed, and acceleration at time t:

$\vec{r}'(t) = \langle 2, 6t \rangle$ = velocity $\vec{v}(t)$

$|\vec{v}(t)| = \sqrt{4 + 36t^2}$ = speed

$\vec{a}(t) = \vec{r}''(t) = \langle 0, 6 \rangle$ = acceleration

Ex. If $\vec{r}(t) = (t^2+1)\vec{i} - t^3\vec{j} + 2t^2\vec{k}$

is the position of a particle,

compute the velocity vector, the speed, and the acceleration

$$\vec{v}(t) = \vec{r}\,'(t) = 2t\vec{i} - 3t^2\vec{j} + 4t\vec{k}$$

$$\therefore \text{Speed} = \sqrt{4t^2 + 9t^4 + 16t^2}$$

$$= \sqrt{20t^2 + 9t^4}$$

$$\vec{a}(t) = \vec{r}\,''(t) = 2\vec{i} - 6t\vec{j} + 4\vec{k}$$

is the acceleration

We can go in the other direction and assume we know the acceleration:

Ex. Suppose the acceleration of a particle is

$$\vec{a}(t) = (2t-1)\vec{i} + 3t^2\vec{j} + (2-t)\vec{k}$$

and also that $\vec{v}(0) = \vec{i} + 2\vec{j}$

These are and $\vec{r}(0) = \vec{j} - \vec{k}$

the initial conditions.

Find the position $\vec{r}(t)$.

$$\vec{v}'(t) = \vec{a}(t), \quad \text{so } \vec{v}(t) = \int \vec{a}(t)\,dt$$

Hence $\vec{v}(t) = (t^2 - t)\vec{i} + t^3\vec{j} + \left(2t - \frac{t^2}{2}\right)\vec{k}$

$+ \vec{C}$

$\vec{v}(0) = \vec{0} + \vec{C}$

$\therefore \vec{C} = \vec{i} + 2\vec{j}$

$\rightarrow \vec{v}(t) = (t^2 - t + 1)\vec{i} + (t^3 + 2)\vec{j}$

$+ \left(2t - \frac{t^2}{2}\right)\vec{k}$

Similarly,

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\frac{t^4}{4} + 2t\right)$$

$$+ \left(t^2 - \frac{t^3}{6}\right)\vec{k} + \vec{C}$$

$\vec{C} = \vec{r}(0) = \vec{j} - \vec{k}$, so

$$\vec{r}(t) = \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\frac{t^4}{4} + 2t + 1\right)\vec{j}$$

$$+ \left(t^2 - \frac{t^3}{6} - 1\right)\vec{k}$$

In general, many problems in physics start with the force $\vec{F}$ known. Since $\vec{F} = m\vec{a}$, this leads to knowing $\vec{a}$.

Ex Suppose $\vec{r}(t) = a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}$

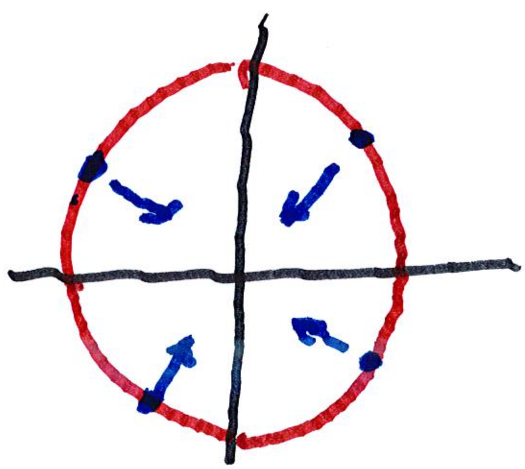
Find $\vec{a}(t)$.

$$\vec{r}\,'(t) = -a\omega\sin\omega t\,\vec{i} + a\omega\cos\omega t\,\vec{j}$$

Hence .

$$\vec{a}(t) = -a\omega^2 \cos\omega t\, \vec{i} - a\omega^2 \sin\omega t\, \vec{j}$$

$$= -\omega^2 (a\cos\omega t\, \vec{i} + a\sin\omega t\, \vec{j})$$



we see that

$$\vec{a}(t) = -\omega^2 \vec{r}(t)$$

This indicates that force acting on the particle points inward.

Motion on earth.

Suppose a projectile is fired

at $t=0$, with $\vec{r}(0) = \vec{0}$

It's initial velocity is

(1) $\vec{v}(0) = V_0 (\cos\alpha \, \vec{i} + \sin\alpha \, \vec{j})$

The force acting on the particle

is $\vec{F} = m\vec{a} = -mg\vec{j}$

$\therefore$ acceleration $= \vec{a} = -g\vec{j}$

$$\rightarrow \vec{v}(t) = -gt\vec{j} + \vec{C} = -gt\vec{j} + \vec{v}(0)$$

$$\rightarrow \vec{r}(t) = \frac{-gt^2}{2}\vec{j} + t\vec{v}(0) + \vec{D}.$$

Since $\vec{r}(0) = \vec{0}$, $\vec{D} = 0$.

Recall (1), we get

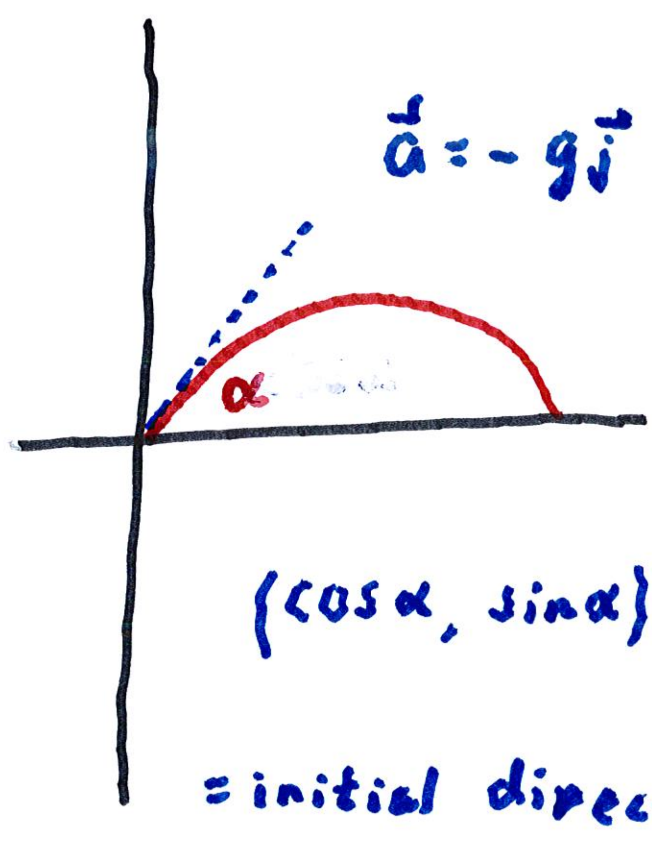
$$\vec{r}(t) = V_0 t \cos\alpha\, \vec{i} + \left(V_0 t \sin\alpha - \frac{gt^2}{2} \right) \vec{j}$$

We obtain

$$\boxed{\begin{aligned} x(t) &= V_0 t \cos\alpha \, \vec{i} \\ y(t) &= \left(V_0 t \sin\alpha - \frac{gt^2}{2} \right) \vec{j} \end{aligned}}$$

Ex. 1 Re

Motion in the plane.



$(\cos\alpha, \sin\alpha)$

$=$ initial direction

Review Problem

Suppose $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$$\kappa = \text{Curvature} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}\,' = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= 6t^2\vec{i} - 6t\vec{j} + 2\vec{k}$$

$$\left| \quad '' \qquad\qquad '' \quad \right| = \sqrt{36t^4 + 36t^2 + 4}$$

$$|\vec{r}\,'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\therefore K = \frac{\sqrt{36t^4 + 36t^2 + 9}}{\left(1 + 4t^2 + 9t^4\right)^{3/2}}$$

A circle of radius a has

$$K = \frac{1}{a}$$

$$K = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{T'(t)}{\vec{r}'(t)} \right|$$

$$K = \frac{|T'(t)|}{|r'(t)|}$$

Ex. Find length of $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$

for $0 \le t \le 1$

$$\vec{r}'(t) = \langle 0, 2t, 3t^2 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4t^2 + 9t^4}$$

$$L = \int_0^1 \sqrt{4t^2 + 36t^4}\, dt$$

$$= \int_0^1 \sqrt{4 + 36t^2}\; t\, dt$$

$u = 4 + 36t^2 \quad u(0) = 4 \quad u(1) = 40$

$du = 72t\,dt$

$\therefore t\,dt = \frac{1}{72}\,du$

$$\int_0^1 = \int_4^{40} \sqrt{u} \cdot \frac{du}{72}$$

$$= \frac{2}{3 \cdot 72}\, u^{3/2} \Big|_4^{40}$$

$$= \frac{1}{108}\left(40^{3/2} - 8\right)$$

Ex. Reparameterize wrt arclength

$$\vec{r}(t) = 2t\,\vec{i} + (1-3t)\,\vec{j} + (\quad +4t)\,\vec{k}$$

$$\vec{r}\,'(t) = \langle 2, -3, 4 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4+9+16} = \sqrt{29}$$

$$\therefore\ s(t) = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$$

$$\Rightarrow \quad s = \sqrt{29}\, t$$

$$\text{or} : \quad t = \frac{s}{\sqrt{29}}$$

$$\vec{r}\left(t(s)\right) = \left\langle \frac{2s}{\sqrt{29}},\ \left(1 - \frac{3s}{\sqrt{29}}\right),\ 5 + \frac{4s}{\sqrt{29}} \right\rangle$$

$$= s\left\langle \frac{2}{\sqrt{29}},\ \frac{-3}{\sqrt{29}},\ \frac{4}{\sqrt{29}} \right\rangle$$

$$+ \langle 0, 1, 5 \rangle$$