14.1 Functions of Several Variables

We often think of a surface

as $z = x^2 y$ or $z = y + x^2$

and we write

$f(x,y) = x^2 y$ or $g(x,y) = y + x^2$.

Instead of 1 variable x,

$f(x,y)$ depends on 2 variables

Def'n A function f of two variables is a rule that assigns to each ordered pair of numbers (x,y) in a set D a unique real number denoted by $f(x,y)$. D is the domain of f and its range is the set of values that f takes, i.e.,

$$R_f = \{ f(x,y) \mid (x,y) \in D \}$$

If the domain D is not specified, then the domain is the set of (x,y) such that $f(x,y)$ is well-defined.

Ex. Find the domain and range of $f(x,y) = \dfrac{\ln(x^2+y-2)}{x-2}$

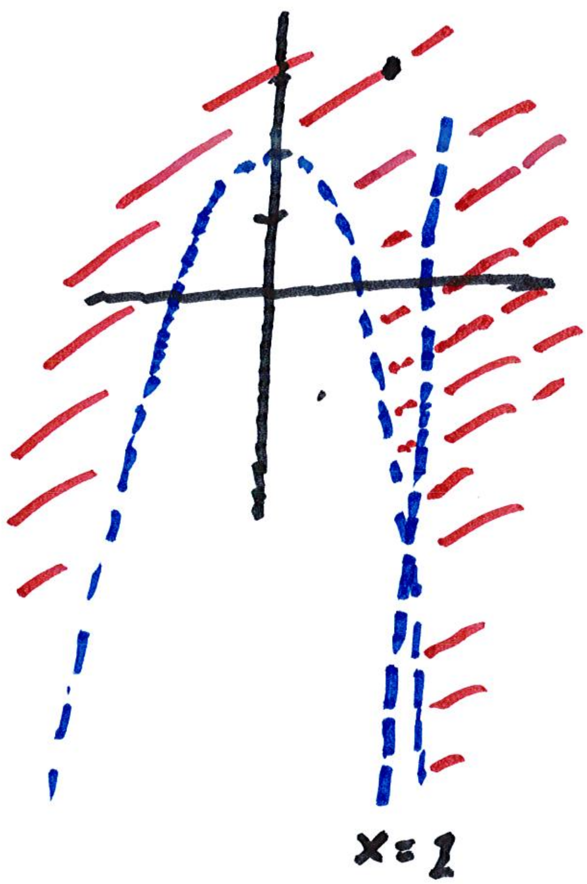
For $\ln(x^2+y-2)$, we need

$x^2 + y - 2 > 0$, i.e.,

$y > 2 - x^2$.

For the denominator $x-2$,

we need $x - 2 \neq 0$ or $x \neq 2$.



$D: \{(x,y) \mid y > 2 - x^2$ and $x \neq 2\}$

Look at $x=0$ and $y > 2$

$$f(0,y) = \frac{\ln(y-2)}{-2}$$ takes on all values.

$\therefore R_f = \{\text{all real numbers}\}$

Ex. Let $g(x,y,z)$

$$= x^3y^2z\sqrt{10-x-y-z}$$

Find the domain and range.

We need $10 - x - y - z \geq 0$,

i.e., $\underbrace{x + y + z \leq 10}_{\text{Domain } D}$

For the range, look

at $x = t$, $y = -t$ and $z = 1$

Ex. $P(x,y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$.

We need $x \geq 0$ and $y \geq 0$

$\therefore D = \{(x,y);\ x \geq 0 \text{ and } y \geq 0\}$

Note that

$P(x,1) = x^{1/4} \geq 0$ for all $x \geq 0$

$\therefore R_p = \{x;\ x \geq 0\}$

Ex. If f is a function of

two variables with domain D,

then the <u>graph</u> of f is the

set of all (x, y, z) such that

$z = f(x,y)$ and $(x,y) \in D$.

Ex. Find the

graph of $\longrightarrow$ $z = 2\sqrt{x^2 + y^2}$

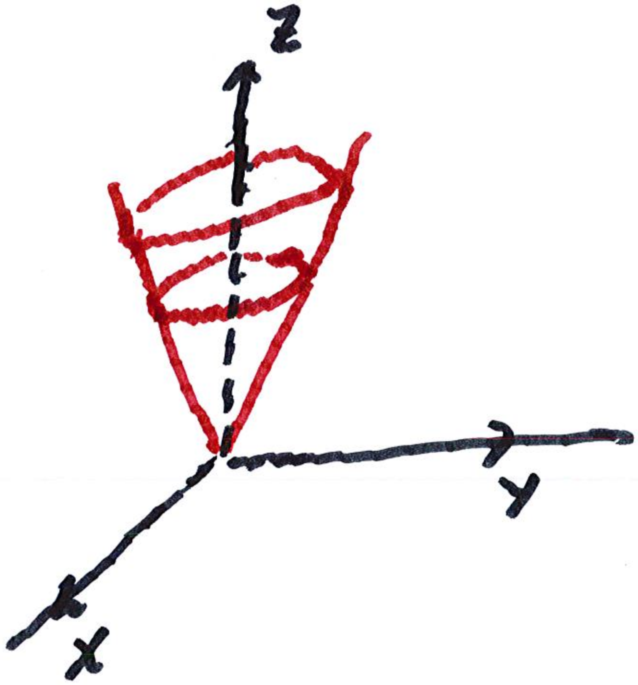
$f(x,y) = 2\sqrt{x^2+y^2}$

or

$z^2 = 4(x^2+y^2)$

$z \geq 0$

any $(x,y) \in$ Dom.



Note that

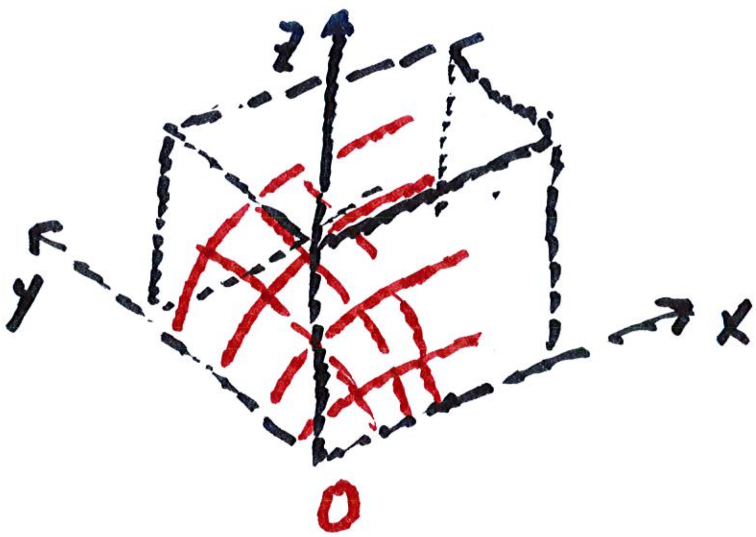
z must be

$z \geq 0$

Sketch the graph of

$f(x, y) = x^{1/4} y^{3/4}$

$z = x^{1/4} y^{3/4}$



Ex. Sketch the graph of

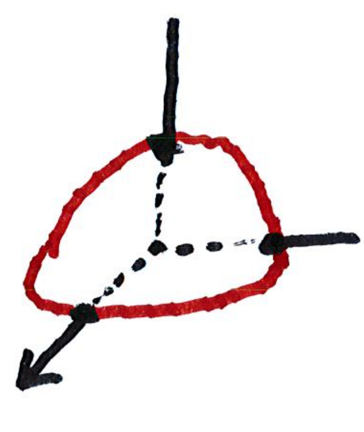
$\sqrt{4-x^2-y^2} \rightarrow z = \sqrt{4-x^2-y^2}$

$\rightarrow z^2 = 4-x^2-y^2$

$\rightarrow x^2+y^2+z^2 = 4$

This is a sphere of radius

2. Recall $z \geq 0$



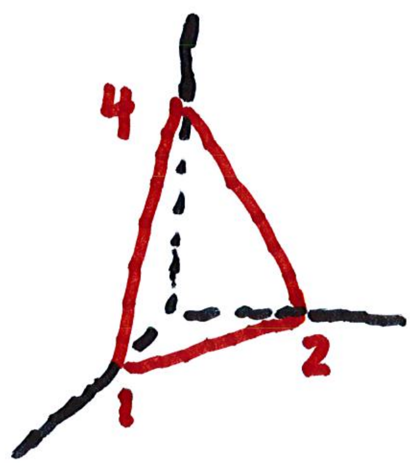
Ex. Find the graph of

$f(x,y) = 4 - 4x - 2y$

$z = 4 - 4x - 2y$

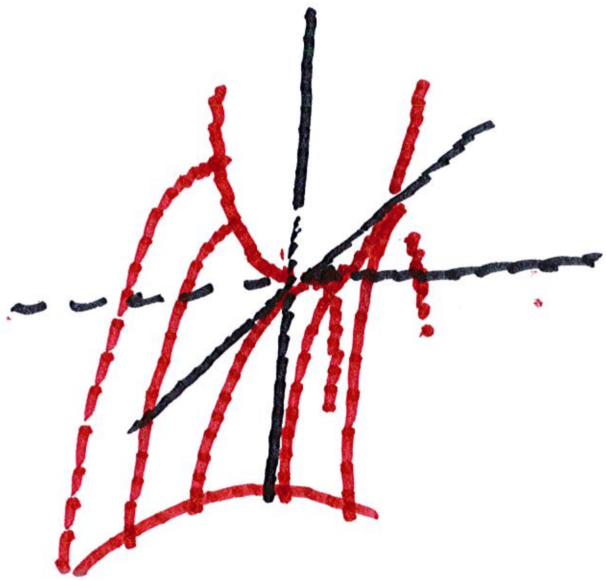
$4x + 2y + z = 4.$

Use x, y, and z intercepts to sketch it



Find Graph of $f(x,y) = y^2 - x^2$

$\rightarrow \quad z = y^2 - x^2$



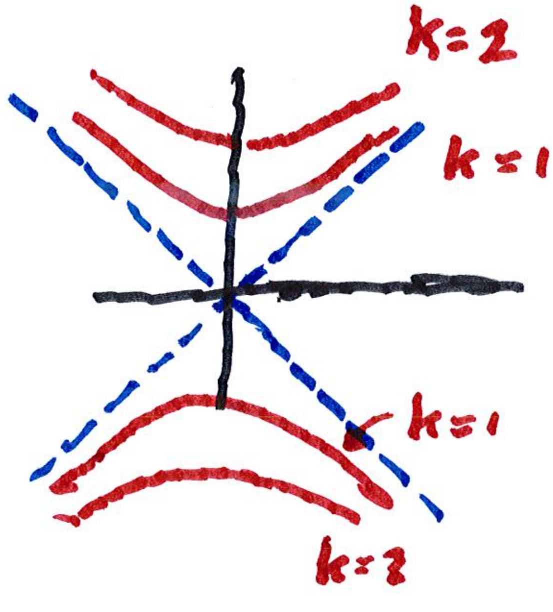
Saddle point at origin.

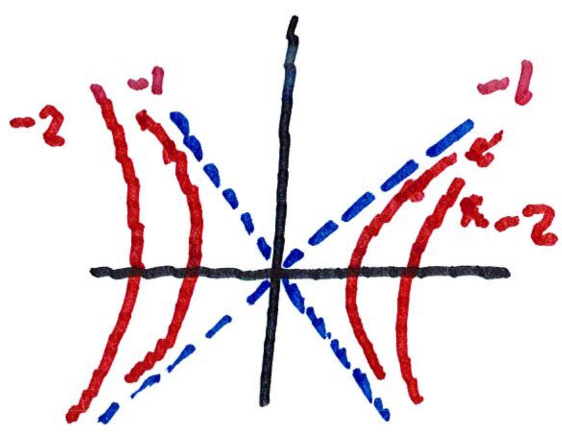
Find the level curves and the graph of $f(x,y) = (1-x-y)^2$

Level Curves of $f(x,y)$

A level curve is $= \{(x,y) \mid f(x,y) = k\}$

Ex. If $f(x,y) = \{x,y\} \mid y^2 - x^2 = k\}$



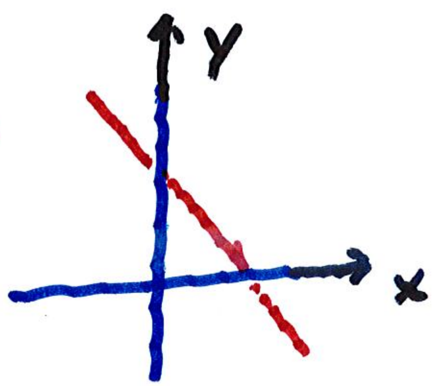


11. Sketch the level curves of $f(x,y) = (x+y-1)^2$

If $k < 0$, $(x+y-1)^2 = k$. NO SOL'N.

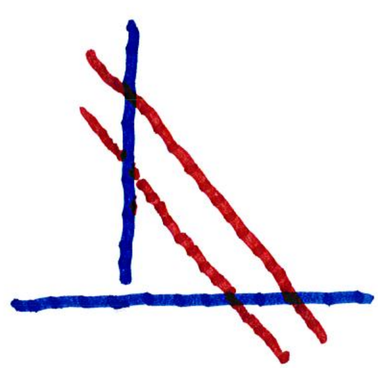
If $k=0$, level curve is

$x+y-1=0$



If $k > 0$, we get $(x+y-1)^2 = k$

$x+y-1 = \pm\sqrt{k}$

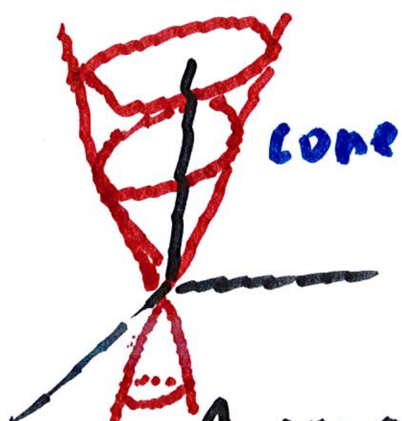


A level surface of a function $f(x,y,z)$ is the set of points (x,y,z) such that $f(x,y,z) = k$.

Sketch the level surface of

$x^2 + y^2 - z^2$

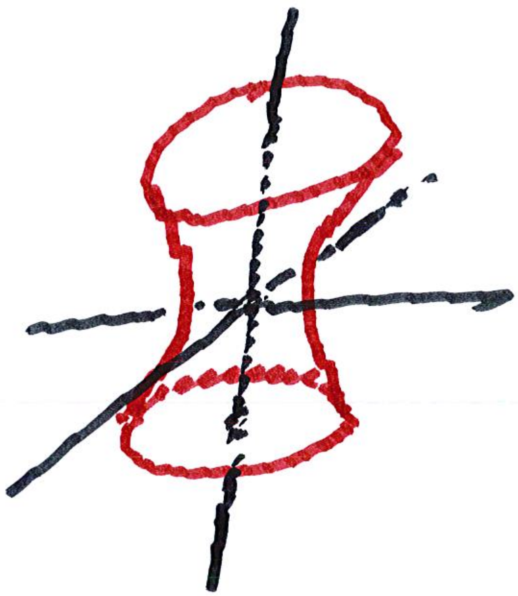
If $k = 0 \rightarrow z^2 = x^2 + y^2$



A cone is rotated about the z-axis

If $k > 0$, say $k = 1$

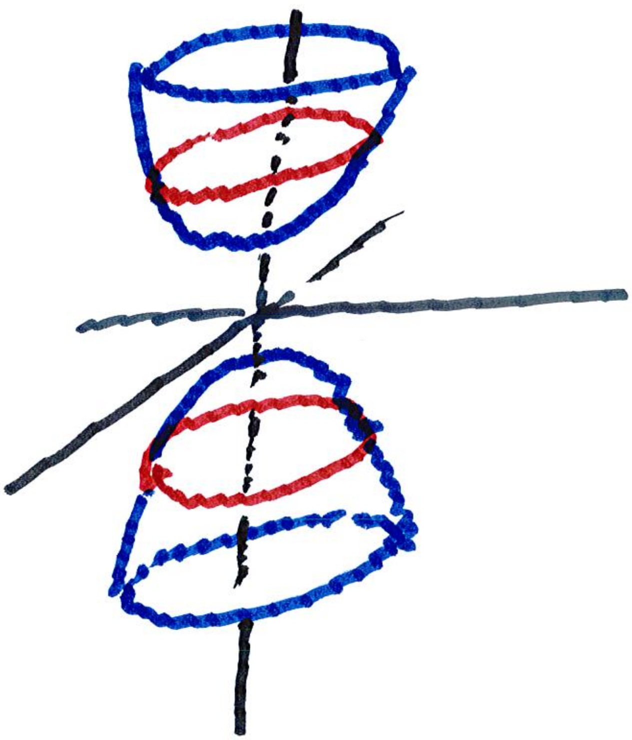
$x^2 + y^2 = 1 + z^2$



hyperboloid of 1 sheet.

Now suppose that $k < 0$,

say $k = -1$



hyperboloid of 2 sheets

Ex. Find the level surfaces

$$\left(\text{The set of } (x,y,z) \;\middle|\; x^2+y^2+z^2=k \right)$$

If $k<0$, no solution at all

If $k=0$, $x^2+y^2+z^2=0$

(the origin)

If $k>0$, $x^2+y^2+z^2=k$

$\rightarrow$ sphere of radius $\sqrt{k}$