Thm 1. If $A \subseteq \mathbb{R}$, let $f: A \longrightarrow \mathbb{R}$ and let c be a cluster point of A. If f has a limit at c, then there are numbers δ and m_0 such that if $x \in A \cap B'_\delta(c)$, then $|f(x)| \leq m_0$.

Proof. Let $\varepsilon = 1$. Then there is a number $\delta_0 > 0$ so that if $x \in A \cap B'_{\delta_0}$, then

$$|f(x) - L| < 1.$$

By the Triangle Property,

$$\begin{aligned} |f(x)| &= |(f(x) - L) + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L| \end{aligned}$$

$\therefore$ Set $m_0 = 1 + |L|$

Thm 2. Suppose that f and g are functions defined on A (except possibly for $x = c$) such that

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M.$$

Then

(i) $\lim_{x \to c} (f+g)(x) = L + M$

(ii) If $b \in \mathbb{R}$, then $\lim_{x \to c} bf(x) = bL$

(iii) $\lim_{x \to c} f(x) g(x) = LM$

(iv) If $g(x) \neq 0$ and $M \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof of (i). Let $\varepsilon > 0$. By the definition of the limit, there numbers δ_1 and $\delta_2 > 0$ such that if $x \in A \cap B'_{\delta_1}(c)$, then

$|f(x) - L| < \frac{\varepsilon}{2}$, and if

$x \in A \cap B'_{\delta_2}(c)$, then

$|g(x) - M| < \frac{\varepsilon}{2}$. Now set

$\delta = \min\{\delta_1, \delta_2\}$. If

$x \in A \cap B'_{\delta}(c)$, then

$|(f(x) + g(x)) - (L + M)|$

$= |(f(x) - L) + (g(x) - M)|$

$$\leq |f(x) - L| + |g(x) - M|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

which proves (i)

Pf. of (iii). Note that

$$|f(x)g(x) - LM|$$

$$= |(f(x) - L)g(x) + (g(x) - M)L|$$

$$\leq |f(x) - L||g(x)| + |g(x) - M||L|$$

By Thm 1, there are constants

m_0 and δ_0 so that

if $x \in A \cap B'_{\delta_0}(c)$, then

$$|g(x)| \le m_0 .$$

Also there are constants

δ_1 and δ_2, so that

$$|f(x) - L| < \frac{\varepsilon}{2m_0}, \quad \text{if } x \in A \cap B'_{\delta_1}(c).$$

and

$$|g(x) - \quad .| < \frac{\varepsilon}{2|L|+1}$$

Now set $\delta = \min\{\delta_0, \delta_1, \delta_2\}$.

If $x \in A \cap B_\delta'(c)$, then

$|f(x)g(x) - LM|$

$\leq \dfrac{\varepsilon}{2m_0} \cdot m_0 + \dfrac{\varepsilon}{2|L|+1} \cdot |L|$

$\leq \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon,$

which proves (iii).

Pf. of (ii). This follows from (iii) by setting $g(x) = b$ for all $x \in A$.

Pf. of (iv). We first show that if $\lim g(x) = M \neq 0$ and if $g(x) \neq 0$, then

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}.$$

The general case follows from (iii) by using the Product Rule.

We need the following:

Proposition. If $\lim_{x \to c} g(x) = M$, and if $M \neq 0$, then there is $\delta_0 > 0$ so that if $x \in A \cap B'_{\delta_0}(c)$, then

$$|g(x)| > \frac{|M|}{2}.$$

Pf. Set $\varepsilon = \frac{|M|}{2}$. Then there is $\delta_0 > 0$ so that

$$|g(x) - M| < \frac{|M|}{2}.$$

$$|g(x)| = |M + (g(x) - M)|$$

$$\geq |M| - |g(x) - M|$$

$$\geq |M| - \frac{|M|}{2} = \frac{|M|}{2}.$$

Now we can prove the Quotient Rule. Since we just showed that

$$\frac{1}{|g(x)|} \leq \frac{2}{|M|} \quad \text{if } x \in A \cap B'_{\delta_0}(c),$$

we get

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right|$$

$$= \left| \frac{M - g(x)}{g(x) M} \right| \leq \frac{2}{|M|^2} |M - g(x)|$$

Let $\varepsilon > 0$. Then there is a $\delta_3 > 0$ so that if $x \in A \cap B'_{\delta_3}(c)$, then $|g(x) - M| < \frac{M^2 \varepsilon}{2}$

Set $\delta = \min\{\delta_0, \delta_3\}$. Then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| \leq \frac{2}{|M|^2} \cdot \frac{M^2 \varepsilon}{2} = \varepsilon$$

This proves (iv).

Ex. Evaluate $\dfrac{2+x}{3-x+}$

Note that $\lim_{x\to 0} x = 0$

$\therefore$ By (i) $\lim (2+x) = 2+0$
$= 2$

and by (iii) $\lim_{x\to 0} x^2 = 0^2 = 0$

and so by (ii), $\lim 3x^2 = 3\cdot 0$
$= 0$

$\therefore$ By (i), $\lim (2+x+3x^2) = 2$

Finally by the Quotient Rule

$$\lim \frac{2+x}{3-x+3x^2} = \frac{2}{3}.$$

As noted above.

$$\lim_{x \to c} x = c,$$

$$\lim_{x \to c} x^2 = c^2$$

$$\vdots$$

$$\lim_{x \to c} x^k = c^k$$

Moreover

$$\lim_{x \to c} a x^k = a c^k .$$

By the Sum Rule,

$$\lim \left(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \right)$$

$$= \left(a_n c^n + \dots + a_0 \right)$$

Thus if $P(x)$ is any polynomial,

then $\lim_{x \to c} P(x) = P(c)$.

and $\lim_{x \to c} Q(x) = Q(c)$

another polynom-ial

and so, if $R(x) = \dfrac{P(x)}{Q(x)}$,

then $\lim_{x \to c} R(x) = R(c)$,

provided that $Q(c) \neq 0$.