

Combinations of 5.2 Continuous Functions

Recall that a function

$f: A \rightarrow \mathbb{R}$ is continuous

at c if $\lim_{x \rightarrow c} f(x) = f(c)$.

If we define

$$V_\delta(c) = \left\{ x \in \mathbb{R} : |x - c| < \delta \right\}$$

then we can write the

limit as follows:

A function $f: A \rightarrow \mathbb{R}$ is continuous at c if:

For every ε -neighborhood

$V_\varepsilon(f(c))$, there is a

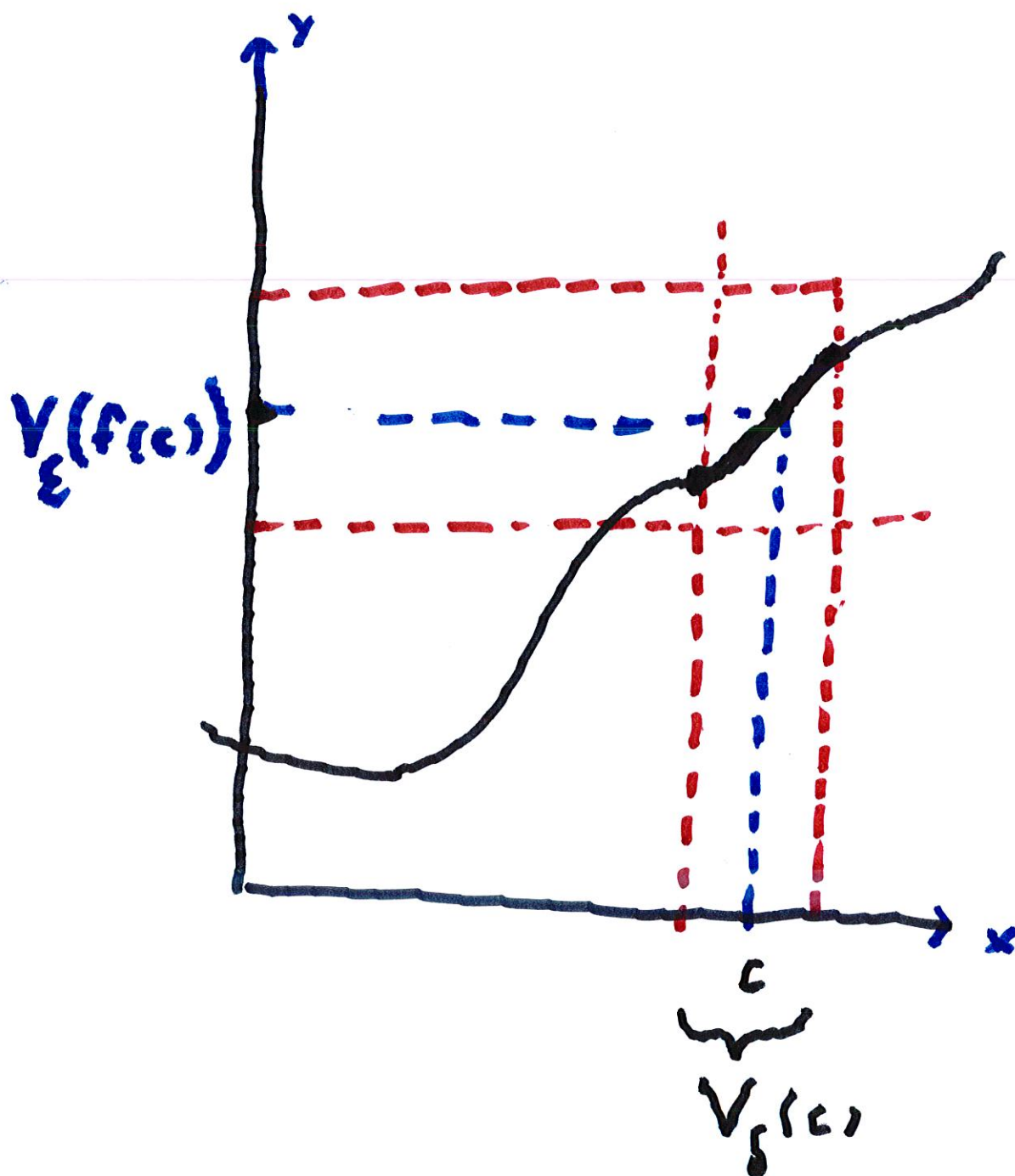
δ -neighborhood $V_\delta(c)$ of c

such that if x is any point

in $V_\delta(c) \cap A$, then $f(x)$

belongs to $V_\varepsilon(f(c))$.

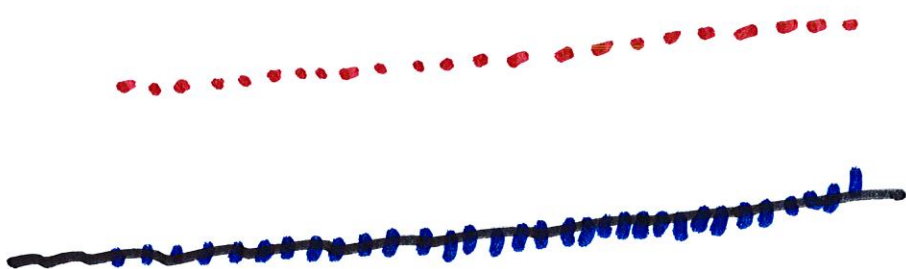
$$\therefore f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)).$$



The y -values of f above $V_\delta(c)$ lie in $V_\epsilon(f(c))$.

Some examples.

$$\text{Set } f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$



f is discontinuous at all x .

If c is irrational, then

since $\mathbb{Q}$ is dense, for every $n > 0$

we can find a rational number

x_n with $|x_n - c| < \frac{1}{n}$ and $x_n \neq c$

If f were continuous at c ,

then $\lim f(x_n) = f(c)$,

i.e. $\lim 1 = 0$, contradiction.

Similarly suppose c is rational.

Since the irrationals

are dense, for every $n > 0$

we can find an irrational d_n

with $0 < |d_n - c| < \frac{1}{n}$.

If f were continuous at c ,

then $\lim_{n \rightarrow \infty} f(a_n) = f(c)$, i.e.

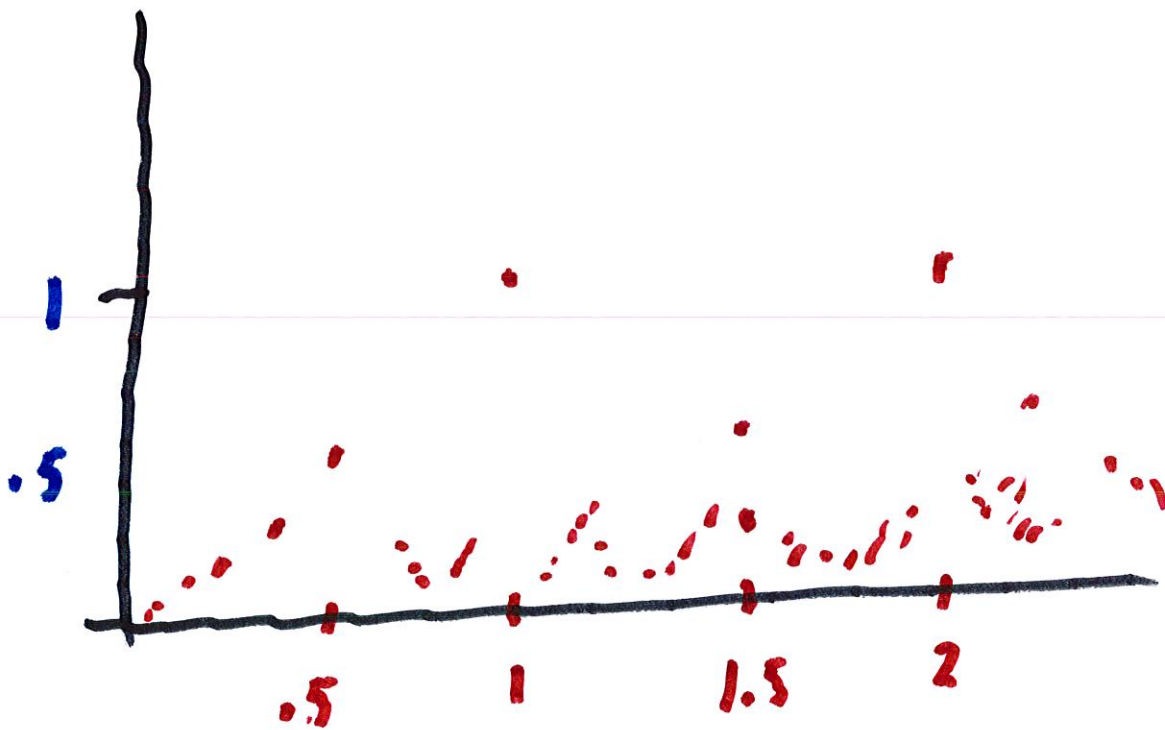
$$\lim_{n \rightarrow \infty} (0) = 1. \text{ Contradiction}$$

Thomae's Fcn.

$$\text{Define } f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

for $x \in [0, 1]$

where p, q have
no common factor



There is a good picture

on pg. 127

First we show that f is
discontinuous at all rational
numbers.

As in the previous example,

let $c = \frac{p}{q}$, so that $f(c) = \frac{1}{q}$.

The irrational numbers are
dense so we choose a sequence
of irrational numbers

x_n such that

$$0 < |x_n - c| < \frac{1}{n}.$$

If f were continuous at c ,

then $\lim f(x_n) = f(c) = \frac{1}{9}$

or $\lim 0 = \frac{1}{9}.$

$\therefore f$ is discontinuous at every rational number.

Now we show that f is
continuous at every

irrational number c . We
can assume that $0 < c < 1$.

Let $\epsilon > 0$, and choose N to
be an integer with $\frac{1}{N} < \epsilon$.

There are only a finite
number of rational numbers

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π_n in $(0, 1)$ such that

$$\pi_n = \frac{p}{q} \text{ and } q \leq N.$$

Choose $\delta > 0$ so that

all of the numbers π_n

lie outside the interval

$I = (c - \delta, c + \delta)$. Thus the

only rational numbers π

in I all have denominators

with $q \geq N$. It follows

that for $|x - c| < \delta$, we have

$$|f(x) - f(c)|$$

$$= |f(x)| = f(x) \leq \frac{1}{N} < \epsilon.$$

It follows that if

$$0 < |x - c| < \delta,$$

$$\text{then } |f(x) - f(c)| < \epsilon.$$

Thus f is continuous at each
irrational c .

5.2 Combinations of Continuous Functions

Recall that if

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M.$$

then

$$1. \lim (f + g) = L + M$$

$$2. \lim (f - g) = L - M$$

$$3. \lim (fg) = LM \quad 4. \lim bf = bL$$

$$5. \text{ if } M \neq 0, \text{ then } \lim f/g = \frac{L}{M}$$

When f and g are continuous at c , then

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{and}$$

$$\lim_{x \rightarrow c} g(x) = g(c). \quad \text{Hence,}$$

$$1. \lim_{x \rightarrow c} (f+g) = f(c) + g(c)$$

$$2. \lim_{x \rightarrow c} (f-g) = f(c) - g(c)$$

$$3. \lim_{x \rightarrow c} (fg) = f(c)g(c)$$

$$4. \lim_{x \rightarrow c} (bf) = b f(c)$$

5. If $g(c) \neq 0$, then

$$\lim_{x \rightarrow c} (f/g) = f(c)/g(c)$$

This implies that $f+g$.

$f-g$, fg , bf and f/g are

all continuous at c .

(provided that
 $g(c) \neq 0$ in 5.)

It follows that any polynomial
and also every rational

$$\text{function } R(x) = P(x)/Q(x)$$

are continuous at every c (except
when $Q(x) = 0$)

We say a function f defined
on A is continuous
on A if f is continuous
at each $c \in A$.

Composition of Continuous Fcns.

Suppose $f: A \rightarrow \mathbb{R}$ is continuous at c

and that $g: B \rightarrow \mathbb{R}$ is continuous at $b = g(c)$,

then we'll show

$(g \circ f)(x) = g(f(x))$ is also

continuous at c , provided

$$f(A) \subseteq B.$$

More precisely:

Thm. Let $A, B \subseteq \mathbb{R}$ and

let $f: A \rightarrow \mathbb{R}$ and

$g: B \rightarrow \mathbb{R}$

be functions such that

$f(A) \subseteq B$. If f is continuous
at a point $c \in A$ and g is

continuous at $b = f(c) \in B$, then

the composition $g \circ f$ is
continuous at c .

Proof: Let W_ξ be an ξ -neighbor-
hood

of $g(b)$. Since g is continuous

at b , there is a δ -neighborhood

V_δ of $b = f(c)$ such that if $y \in B \cap V$

then $g(y) \in W$. Since f is

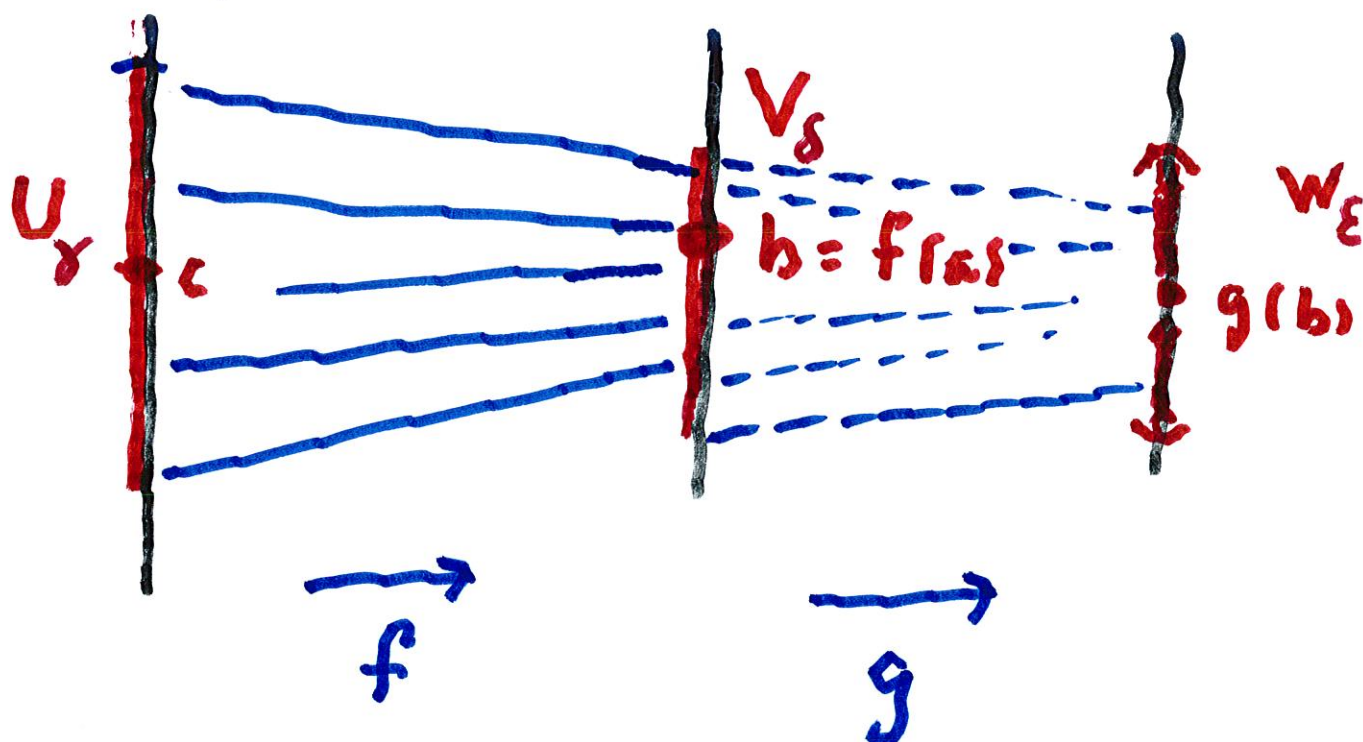
continuous at c , there is a Y -neighborhood U_y of c such that if $x \in A \cap V$, then $f(x) \in V$. Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$, then $f(x) \in B \cap V$ so that $(g \circ f)(x) = g(f(x)) \in W$. But

since W is an arbitrary

ϵ -neighborhood of $g(b)$, this

implies that $g \circ f$ is continuous

at c .



Thm

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Let $f: A \rightarrow \mathbb{R}$ and

let $g: B \rightarrow \mathbb{R}$

be continuous on A and

B respectively. If

$f(A) \subseteq B$, then

$g \circ f: A \rightarrow \mathbb{R}$ is continuous

on A .