6.1 The derivative

Def'n. Let $I \subseteq \mathbb{R}$, let $f: I \to \mathbb{R}$ and let $c \in I$. We say f is differentiable at c if there is a number L such that

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

We write $L = f'(c)$, and we say L is the derivative of f at c.

Note that the above definition allows c to be an endpoint of I. If we set $E(x) = (x-c)\,e(x)$,

(where $\dfrac{f(x)-f(c)}{x-c} = L + e(x)$)

then

$$f(x) - f(c) = L(x-c) + E(x)$$

$$\rightarrow \quad f(x) = f(c) + L(x-c) + E(x).$$

Note that

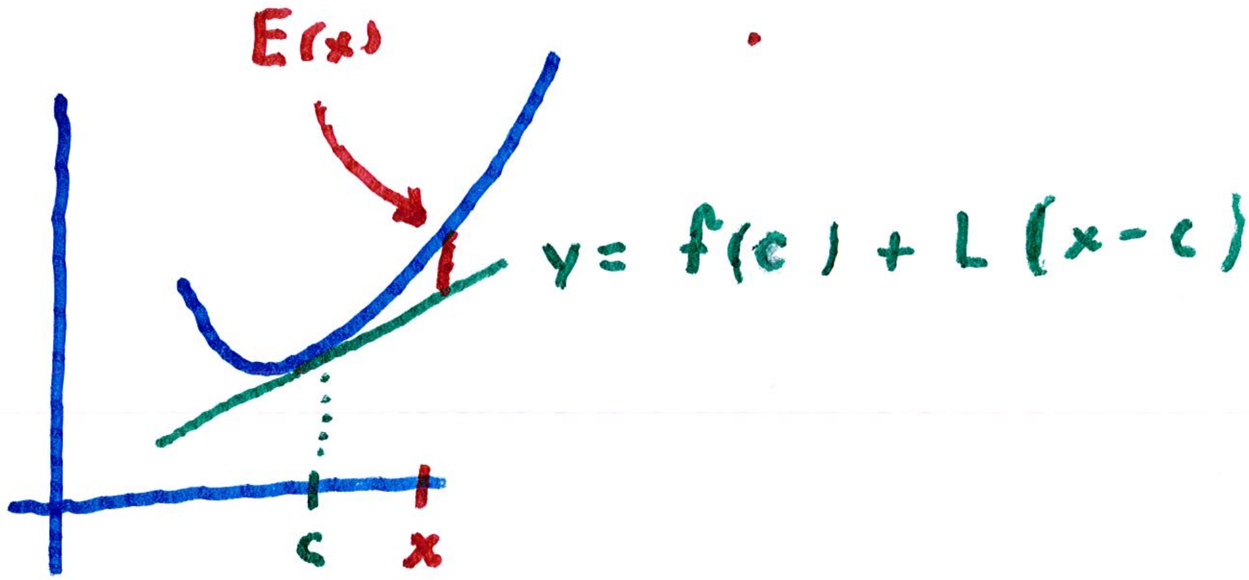
$$\lim_{x\to c} \frac{|E(x)|}{|x-c|} = 0.$$

Thus, $f(x)$ differs from $f(c) + L(x-c)$ by an error $E(x)$ that converges to 0 faster than $|x-c|$.

Conversely, suppose

$$f(x) - \big(f(c) + M(x-c)\big) = E(x)$$

where $\lim_{x \to} \frac{|E(x)|}{|x-c|} = 0$, Then


E(x)
y = f(c) + L(x - c)
c
x

$$\left| \frac{f(x) - f(c)}{x - c} - M \right| = \left| \frac{E(x)}{x - c} \right| .$$

Then $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = M$,

which implies that f is differentiable at c. By the uniqueness of limits,

$M = L$.

Thm. If f is differentiable at $c \in I$, then f is continuous.

Pf. For all $x \in I$,

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c).$$

Taking the limit as $x \to c$, the right-hand side approaches $L \cdot (0) = 0$, which shows that $\lim_{x \to c} f(x) = f(c)$.

which implies that f is continuous at c.

We can show that there analogs of the limit rules for derivatives.

Thm. Suppose that $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are differentiable at c. Then

(i) If $\alpha \in \mathbb{R}$, then αf is differentiable at c, and

$\lim (\alpha f)'(c) = \alpha f'(c)$

(ii) $f+g$ is differentiable at c, and $(f+g)'(c) = f'(c) + g'(c)$

(iii) fg is differentiable at c, and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv). if $g(c) \neq 0$, then f/g is differentiable at c, and

$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\{g(c)\}^2}$$

Proof of (iii) and (iv).

(iii) Let $p(x) = f(x)g(x)$. Then if $x \neq c$ and $x \in I$,

$$\frac{p(x) - p(c)}{x-c} =$$

$$= f(x)g(x) - f(c)g(c)$$

$$= \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x) - f(c)}{x-c}\, g(x) + f(c) \left(\frac{g(x) - g(c)}{x-c} \right).$$

Since f and g are differentiable at c, we have $\lim_{x \to c} g(x) = g(c)$.

Letting x approach c, we get

$$f'(c)\, g(x) + f(c)\, g'(c)$$

Thus, we have proved

$$\lim \frac{p(x) - p(c)}{x - c} = f'(c)g(c) + f(c)g'(c).$$

This proves (iii)

(iv) Let $q = f/g$. Since

g is differentiable at c,

$\lim g(x) = g(c)$. Moreover

since $g(c) \neq 0$, there is an

interval $J \subseteq I$ so that $g(x) \neq 0$

when $x \in J$.

For all $x \in J$ with $x \neq c$,

$$\frac{q(x)-q(c)}{x-c}$$

$$= \frac{f(x)/g(x) - f(c)/g(c)}{x-c}$$

$$= \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(x)g(c)}\left\{\frac{f(x)-f(c)}{x-c}\,g(c) - f(c)\,\frac{g(x)-g(c)}{x-c}\right\}$$

which converges to

$$= \frac{1}{g(c)^2}\left\{f'(c)g(c) - f(c)g'(c)\right\}$$

Thus, we have proved (iv).

Corollary. If $f_1, \ldots, f_n$ are differentiable at c, then $f_1 + \ldots + f_n$ is differentiable at c and

$$(f_1 + \dots + f_n)'(c)$$

$$= f_1'(c) + \dots + f_n'(c),$$ and

$$(f_1 \cdots f_n)'(c)$$

$$= f_1'(c)\big(f_2(c) \cdots f_n(c)\big)$$

$$+ f_1(c) f_2'(c) \cdots f_n(c)$$

$$+$$
$$\vdots$$
$$= f_1(c) f_2(c) \cdots f_n'(c).$$

If $f_1 = \ldots = f_n = f$, then

$$(f^n)'(c) = n\{f(c)\}^{n-1} f'(x).$$

The Chain Rule.

Let I, J be intervals in $\mathbb{R}$,

let $g: I \to \mathbb{R}$, and

let $f: J \to \mathbb{R}$ be functions

such that $f(J) \subseteq I$. Suppose

f is differentiable at c

and g is differentiable at $f(c)$,

then the composition $g \circ f$

is differentiable at c, and

$$(g \circ f)'(c) = g'(f(c))\, f'(c).$$

Pf. We consider two cases:

(i) $f'(x_0) \neq 0$. Then there

are positive numbers

m_1 and m_2 so that

$$m_1 < \left| \frac{f(x) - f(c)}{x - c} \right| < m_2$$

Thus, as x approaches 0,

$f(x) - f(c) \neq 0$, (if $0 < |x-c| < \delta_0$)

Also, $\lim_{x \to c} f(x) = f(c)$

Hence $\lim_{x \to c} \dfrac{g(f(x)) - g(f(c))}{f(x) - f(c)}$

$= g'(f(c))$

This means that

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

$$= g'(f(c)) \cdot f'(c).$$

This proves the Chain Rule

when $f'(c) \neq 0$.

(ii) Now we consider the

case when $f'(c) = 0$.

In this case we need

to prove that

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = 0.$$

If we set $L = 0$ in the definition of the derivative, we get

$$f(x) - f(c) = E(x).$$

But $\lim\limits_{x \to c} \dfrac{|E(x)|}{|x-c|} = 0$, which means that if $|E(x)| < \varepsilon|x-c|$ if $|x-c|$ is small.

On the other hand,

$$\lim \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a)),$$

so $|g(y) - g(f(a))| \le M|y - f(a)|$

if $M > |g'(f(a))|$.

Now set $y = f(x)$. Then

$$|g(f(x)) - g(f(c))| \le M|f(x) - f(c)|$$

$$\le M\varepsilon|x - c|.$$

Since this is true for all small

ε, it follows that

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = 0$$

which $(g \circ f)'(c) = 0$

when $f'(c) = 0$.