Basic Facts about the Darboux Integral.

If f is Darboux integrable we write $f \in D[a,b]$.

Thm. Suppose that f and g are in $D[a,b]$. Then

(a) If $k \in \mathbb{R}$, the function

kf is in $D[a,b]$ and

$$\int_a^b kf = k\int_a^b f .$$

(b) The function $f+g$ is in $D[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g .$$

(c) If $f(x) \le g(x)$, for all $x \in [a,b]$ then $\int_a^b f \le \int_a^b g$.

(d) If $f \in D[a,b]$, then

$|f|$ is in $D[a,b]$, and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \leq M(b-a),$$

where $|f(x)| \leq M$ for all $x \in [a,b]$

(e) The product fg is in $D[a,b]$

The following is more difficult.

The Composition Theorem

Let $f \in D[a,b]$ with

$f([a,b]) \subseteq [c,d]$ and

let φ be a continuous

function from $[c,d]$ to $\mathbb{R}$.

Then the composition $\varphi \circ f$

belongs to $D[a,b]$.

The Additivity Theorem

Let $f : [a,b] \to \mathbb{R}$ and $f \in \mathcal{D}[a,b]$, and let $c \in [a,b)$. Then its restrictions to $[a,c]$ and $[c,b]$ are both in $\mathcal{D}[a,c]$ and $\mathcal{D}[c,d]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

The question arises,

when is a function f in $[a,b]$.

Definition (a) A set $Z \subset \mathbb{R}$ is a **null set** if for every $\varepsilon > 0$, there is a collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and}$$

$$\sum_{k=1}^{\infty} (b_k - a_k).$$

Example. The set of all rational numbers in $[a, b]$ is a null set. In fact define $J_k = \left(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}} \right)$

Clearly, the union $\bigcup_{k=1}^{\infty} J_k$

of these intervals contains

every rational point in $[a,b]$.

The sum of the lengths of

the intervals is $\sum_{k=1}^{\infty} (\varepsilon/2^k) = \varepsilon$.

Theorem (Lebesgue) A

bounded function $f:[a,b] \to \mathbb{R}$

is integrable if and only if the set of discontinuities is a null set.

Fundamental Theorem of Calculus. (First Form). Suppose there is a finite set E in $[a, b]$, and that these functions satisfy

(a) F is continuous on $[a,b]$.

(b) $F'(x) = f(x)$ for all x in $[a,b] \setminus E$.

(c) f belongs to $D[a,b]$.

Then we have

$$\int_a^b f = F(b) - F(a).$$

In the case when E is the empty case, we say the function F is an <u>antiderivative of f</u>, if F is continuous at all $x \in [a, b]$ and if f is differentiable at each point of (a, b) and

$$F'(x) = f(x) \quad \text{for all } x \in (a, b)$$

Then one has the same

conclusion:

$$\int_a^b f = F(b) - F(a). \qquad (1)$$

In other words, to compute

$\int_a^b f$, we just have to find

an antiderivative F on $[a,b]$

and apply (1).

Proof. We prove the above theorem in the case where $E : \{a, b\}$. The general case can be obtained by breaking $[a, b]$ into a finite set of intervals.

The Mean Value Theorem