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Spring Semester 2017

MA 341 web Page

Lectures

Homework

1.3 Finite and Infinite Sets

Let $N_m = \{1, 2, \ldots, m\}$.

1. A set S has <u>m elements</u> if there is a <u>bijection</u> f from N_m onto S

2. A set S is <u>finite</u> if it has m elements (m is unique).

3. S is <u>infinite</u> if it is not finite

4. A set S is <u>denumerable</u> if there is a bijection of N onto S

5. S is <u>countable</u> if it is either finite or denumerable.

6. S is <u>uncountable</u> if it is not countable

Ex. Some examples.

The set $E = \{2n : n \in \mathbb{N}\}$

of even natural numbers

is denumerable.

So is $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$

So is $P = \{2, 3, 5, 7, 11, \dots\}$

(the set of prime numbers).

$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

$f(n) = \frac{n}{2}$ if n is even

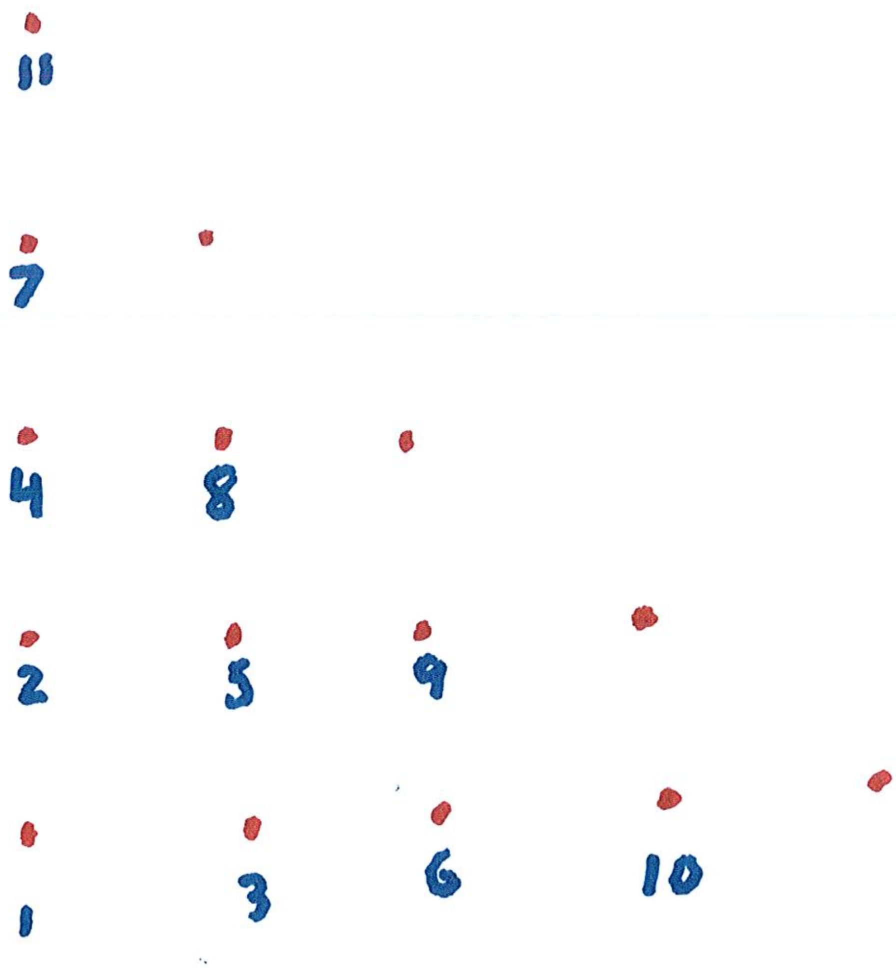
$f(n) = -\frac{n-1}{2}$ if n is odd.

is the formula for the bijection of N onto $\mathbb{Z}$.

Is $N \times N$ denumerable?

(1, 4)			
(1, 3)	(2, 3)	(3, 3)	
(1, 2)	(2, 2)	(3, 2)	
(1, 1)	(2, 1)	(3, 1)	(4, 1)

Fallow first diagonal, then the second, then the third, etc. .


11
7
4
8
2
5
9
1
3
6
10

Using this method, let

$f(m,n)$ = value assigned

to (m,n).

Thus $f(1,1) = 1$ $f(1,2) = 2$

$f(2,1) = 3$. $f(1,3) = 4$

... $f(4,1) = 10$, ...

Sum of first 2 diagonals

$= 1 + 2 = 3$ $f(2,1) = 3$

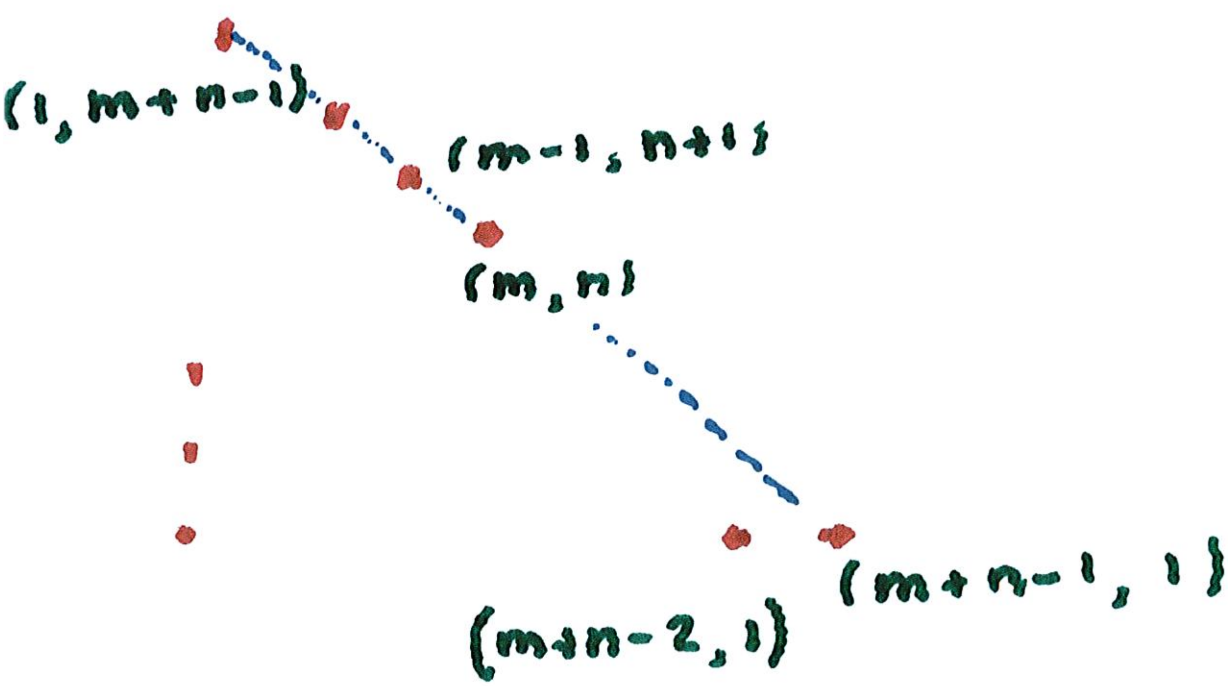
Sum of k diagonals is

$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$.

$f(k,1) = \frac{k(k+1)}{2}$.

We see that the endpoints of $(m+n-1)$-th diagonal are $\underline{(1, m+n-1)}$ and $\underline{(m+n-1, 1)}$.

Hence the predecessor of $(1, m+n-1)$ is $(1, m+n-2)$.



Hence,

$$f(m,n) = f(m-1, n+1) + 1$$

$$= f(m-2, n+2) + 2$$

$$\vdots$$

$$= f(1, m+n-1) + (m-1)$$

$$= f(m+n-2, 1) + m$$

$$f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

Observe that as we move along the path, $f(m,n)$ increases by 1 with each step. Therefore,

$$f: N \times N \to N \text{ is 1-to-1 and onto}$$

It follows that f has an inverse $g: N \to N \times N$ that is also 1-to-1 and onto.

g satisfies

$$g(1) = (1,1)$$

$$g(2) = (1,2)$$

$$g(3) = (2,1)$$

$$g(4) = (1,3), \quad \text{etc.}$$

In general

$$g(k) = (m(k), n(k))$$

for $k = 1, 2, \ldots$

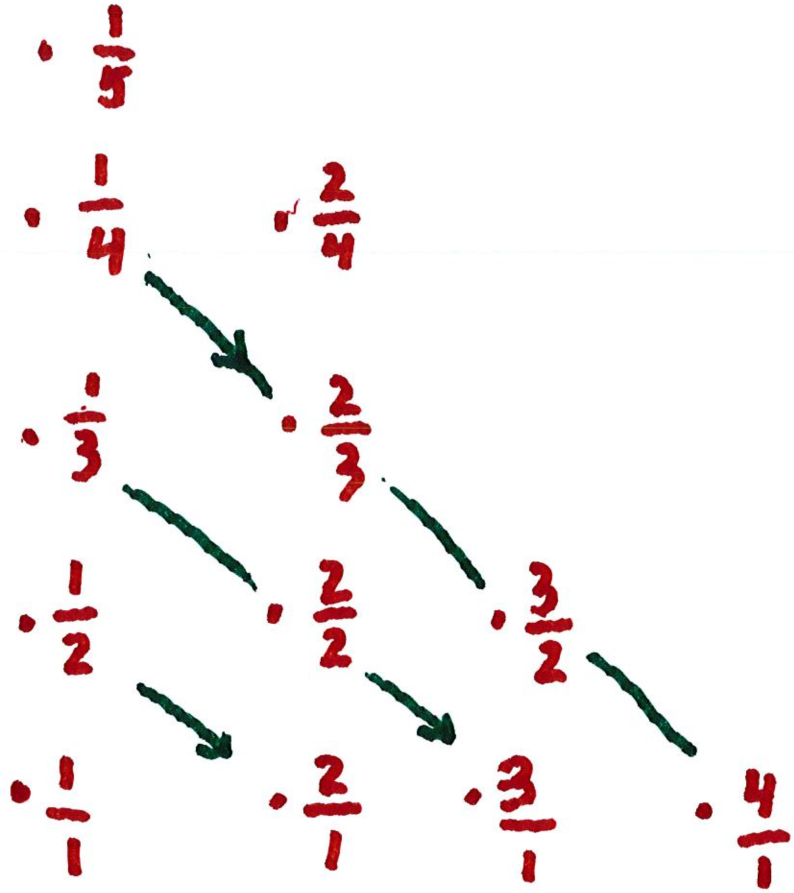
Now define a

function $\pi(m,n) = \frac{m}{n}$

and also define

$$h(k) = \pi(g(k)) = \frac{m(k)}{n(k)}$$

This is the k-th positive rational number at the k-th point on the path.



$h(1) = \frac{1}{1}$ $\quad h(2) = \frac{1}{2}$ $\quad h(3) =$

Thus we obtain a function $h: N \to Q^+$ that is onto but not 1-to-1.

We want to modify h to make it 1-to-1 and onto.

But we first prove:

Thm. 1. Suppose that $h : N \to S$ is surjective, where S is infinite. Then there is a function $H : N \to S$ that is 1-to-1 and onto. Thus, S is denumerable.

Pf. Set $x_n = h(n)$, and

set $n_1 = 1$.

Let x_{n_2} be the smallest

positive integer such that

$x_{n_2} \neq x_{n_1}$ (Hence, if $n_1 < \ell < n_2$

then $x_\ell = n_1$)

Having chosen $n_1, n_2, \ldots, n_{k-1}$,

$k = 3, 4, \ldots$

let n_k be the

smallest integer greater than

n_{k-1} such than

$$x_{n_k} \notin \{n_1, n_2, \ldots n_{k-1}\}$$

(Note that if $n_{k-1} < \ell < n_k$,

Then $x_\ell \in \{x_{n_1}, \ldots, x_{n_{k-1}}\}$)

Now set $H(k) = x_{n_k}$, $k = 1, 2, \ldots$

Then H is a bijection of

$\mathbb{N}$ onto S.

If we apply Thm. 1 to the

sequence $h : \mathbb{N} \to \mathbb{Q}^+$, then H

is a bijection of $\mathbb{N}$ onto $\mathbb{Q}^+$.

Thus we have listed all

the rational numbers by

$r_1, \quad r_2, \quad r_3, \ldots r_n, \ldots$

$H(1) = r_1, \quad H(2) = r_2, \ldots$

If we apply the above,

we see that all pairs of

national numbers can be

written in a list, but not 1-to-1.

We can write $r_k = H(k)$.

Now consider the list of ordered pairs of rational numbers.

$$\begin{array}{llll}
r_1, r_4 & & & \\
r_1, r_3 & r_2, r_3 & r_3, r_3 & \\
r_1, r_2 & r_2, r_2 & r_3, r_2 & \\
r_1, r_1 & r_2, r_1 & r_3, r_1 & r_4, r_1
\end{array}$$

In the usual way we obtain a surjection h of N onto $Q^+ \times Q^+$. Thm. 1 implies there is a bijection H of $N \to Q^+ \times Q^+$

Thus $Q^+ \times Q^+$ is denumerable.

By Induction, the set $Q^+ \times \cdots \times Q^+$ of all p-tuples of rational numbers is denumerable.