Differential Equations.

Recall that in order to solve

the differential equation

$$y'(x) = f(x, y(x)), \quad \text{with } y(x_0) = y_0$$

we set up an iterative scheme:

$$y_{j+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_j(t))\, dt$$

starting with $y_0(x) = y_0$.

The function $f(x, y(x))$ satisfies

$|f(x,y)| \leq M$ for all (x,y) in a rectangl R_h defined by $|x-x_0| \leq h$ and $|y-y_0| \leq Mh$. In addition f satisfies a Lipschitz condition

$$|f(x,s) - f(x,t)| \leq C|s-t|,$$

for all points (x,s) and (x,t) in R_h. Finally h is chosen sufficiently small so that $Ch < 1$. If K and L are positive integers with $K < M$, we obtained the estimate

$$|y_K(x) - y_L(x)| \le \frac{Mh\,(Ch)^K}{1 - Ch}.$$

We want to show that the functions $y_K \to y(x)$, as $K \to \infty$

Definition. A sequence of functions on $A \subseteq \mathbb{R}$ converges uniformly on A to a function $f : A \to \mathbb{R}$ if for each $\varepsilon > 0$ there is an integer $K(\varepsilon)$ (independent of x) such that if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in A.$$

We write $f_n \rightrightarrows f$ on A

Definition. If $\varphi : A \to \mathbb{R}$ is bounded on A, we define the uniform norm of φ by

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}$$

Thus, if $\varepsilon > 0$,

$$\|\varphi\|_A \le \varepsilon \Leftrightarrow |\varphi(x)| \le \varepsilon, \text{ for all } x \in A.$$

Note that a sequence converges uniformly on A to f if and only if

$\|f_n - f\|_A \to 0.$

We can define Cauchy sequences of functions.

Theorem. Let f_n be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly to a bounded function f if and only if for each $\varepsilon > 0$, there is an integer $H(\varepsilon)$ such that for all $m, n > H(\varepsilon)$, then $\|f_m - f_n\| \leq \varepsilon$.

We prove the $\Leftarrow$ direction.

Suppose that for $\varepsilon > 0$, there is $H(\varepsilon) > 0$ such that

if $m, n \geq H(\varepsilon)$, then $\|f_m - f_n\| \leq \varepsilon$.

Thus, for each $x \in A$, we have

$$|f_m(x) - f_n(x)| \le \|f_m - f_n\|_A \le \varepsilon,$$

for $m, n \ge H(\varepsilon)$.

Hence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ which converges to $f(x)$. Therefore,

$f(x) = \lim_{n\to\infty} f_n(x)$ satisfies

$|f_m(x) - f(x)| \le \varepsilon$. for all $x \in A$ and $m \ge$ ~~$k(\varepsilon)$~~ $H(\varepsilon)$

We conclude that (f_n) converges uniformly on A to f

In our differential equations setting, since

$$|y_K(x) - y_L(x)| \leq \frac{Mh\,(Ch)^K}{1-Ch},$$

we conclude that if

$$A = \{x : |x - x_0| \leq h\}.$$

then the sequence of functions (y_K) on A converges uniformly to a bounded function on A.

Now we want to show that the function $y(x)$ is continuous on A.

We have a general theorem:

Theorem. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f: A \to \mathbb{R}$. Then f is continuous on A.

Proof: By hypothesis, given

$\varepsilon > 0$, there exists an integer

$H = H(\varepsilon) \in \mathbb{N}$ such that if

$n \geq H$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

Let $c \in A$ be arbitrary.

By the Triangle Inequality,

we have $|f(x) - f(c)|$

$\leq |f(x) - f_H(x)| + |f_H(x) - f_H(c)|$

$+ |f_H(c) - f(c)| \leq \frac{\varepsilon}{3} + |f_H(x) - f_H(c)| + \frac{\varepsilon}{3}$.

Since f_H is continuous, there exists a number $\delta = \delta(\frac{\varepsilon}{3}, c, f_H) > 0$ such that if $|x-c| < \delta$ and $x \in A$, then $|f_H(x) - f_H(c)| < \frac{\varepsilon}{3}$.

Since $\varepsilon > 0$ is arbitrary, it follows that f is continuous at c.

Back to our differential equations, it follows that the function $y(x) = \lim_{K \to \infty} y_K(x)$

is continuous on A.

Moreover, since each function

$y_K(x)$ satisfies $|y_K - y_0| \leq M$

it follows that $(x, y(x))$

also lies in R_h.

It remains to show that

$y = y(x)$ satisfies the integral

equation.

The function $f(x,$

is continuous in R_n. In particular, f is continuous in a neighborhood of the set $\{(x, y(x)) ; |x-x_0| \leq h.\}$

Since $y_j(x)$ converges uniformly to $y(x)$, we conclude that

$y_{j+1}(x) - y_0 - f(x, y_j(x))$ converges to $y(x) - y_0 - f(x, y(x)) \equiv 0$

for all x with $|x - x_0| \leq h$.

We conclude that

$$y(x) = y_0 + f(x, y(x)),$$

which completes the proof.