Roots of Positive Numbers

Thm. For every real $x > 0$ and every integer $n > 0$, there is one and only one positive real y such that $y^n = x$. This number y is written $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Proof. That there is at most one such y is clear

Since $0 < y_1 < y_2$ implies

$0 < y_1^n < y_2^n$.

Let E be the set of all

positive real numbers t

such that $t^n < x$.

If $t = \frac{x}{1+x}$, then $0 < t < 1$.

Hence $t^n < t < x$. Thus

$t \in E$, and E is not empty.

If $t > 1+x$, then $t^n > t > x$, so that $t \notin E$. Thus $1+x$ is an upper bound of E.

The Least Upper Bound Property implies that there is $y = \text{l.u.b. } E$

To prove that $y^n = x$, we show that each of the inequalities $y^n < x$ and $y^n \quad x$ leads to a contradiction.

The identity

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

yields the inequality

$$b^n - a^n < (b-a)\,n\,b^{n-1}$$

when $0 < a < b$.

Assume $y^n < x$. Choose

$h > 0$ so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put $a = y$ and $b = y + h$. Then

$$(1) \quad (y+h)^n - y^n < hn(y+h)^{n-1}$$

$$< hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$ and $y+h \in E$.

Since $y+h > y$, this contradicts the fact that y is an upper bound of E.

Assume now that $y^n > x$.

Put $k = \dfrac{y^n - x}{n y^{n-1}}$

Then $0 < k < y$. The above identity (1) becomes

$y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus $(y-k)^n > x$ and $y-k \notin E$.

Moreover, if $t \geq y-k$, then

$t^n \geq (y-k)^n > x$. It follows

that $y-k$ is an upper bound

of E, which contradicts

the fact that y is the

least upper bound of E.

It follows that $y^n = x$

2.5 Intervals

We need to prove a theorem

about "nested intervals"

before we study 3.4.

We say a sequence of closed

$\underbrace{\text{bounded}}_{}$ intervals are **nested** if

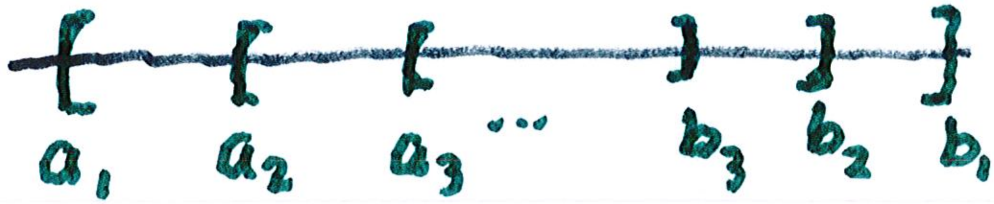
$$I_1 \supseteq I_2 \supseteq \ldots \supset I_n \supset I_{n+1} \supset \ldots$$

If $I_n = [a_n, b_n]$, then

(b_n) is decreasing, and

(a_n) is increasing, i.e.

we have the picture



We prove the

Nested Interval Property:

Given a sequence of nested closed intervals as above, there is a point η in I_n for all $n \in \mathbb{N}$

Proof. Since $I_n \subseteq I_1$,

we get

$a_n \le b_n \le b_1$, for all $n \in \mathbb{N}$.

Hence the sequence (a_n)

is increasing and bounded.

By the Monotone Convergence

Thm., there is an η

satisfying $\eta = \lim(a_n)$.

Clearly $a_n \leq \eta$, all $n \in \mathbb{N}$. (1)

We want to show that

$\eta \leq b_n$ for all n.

We do this by showing that

for any particular n,

$b_n \geq a_k$, $k = 1, 2, \ldots$

There are 2 cases.

(i) If $n \leq k$, then since

$I_n \supseteq I_k$, we have

$a_k \le b_k \le b_n.$

(ii) If $k < n$, then since

$I_k \supseteq I_n$, we have

$a_k \le a_n \le b_n$

We conclude that $a_k \le b_n$.

for all k,

so that b_n is

an upper bound for

$\{a_k ; k \in N\}$

Passing to the limit as

k approaches ∞, we obtain

$\eta \le b_n$, for all $n \in N$. (2)

Combining in (1) and (2),

we have

$a_n \le \eta \le b_n$, all $n \in N$.

Hence $\eta \in I_n$ for all n.

We can use nested intervals to show that the set $\mathbb{R}$ of real numbers is NOT countable.

Suppose that there is a sequence $I = \{x_1, x_2, \dots\}$ such that for any x in $[0,1]$, there is an integer n such that $x_n = x$.

Choose a closed subinterval

$I_1 \subset [0,1]$ such that $x_1 \notin I_1$.

Now choose a closed subinterval

$I_2 \subset I_1$ such that $x_2 \notin I_2$.

In this way we obtain

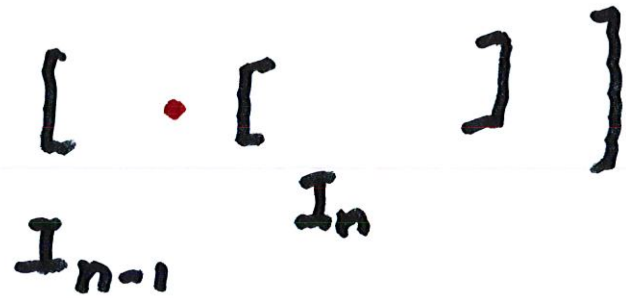
a sequence of closed subintervals

such that

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n$$

such that for all $n = 1, 2, \dots,$

$$x_n \notin I_n$$



The Nested Interval Theorem implies that there is a point $\eta \in I_n$, for all $n = 1, 2, \dots$

Since $x_n \notin I_n$ for all n, it follows that for all $n = 1, 2, \ldots$,

$x_n \neq \eta$.

It follows that $I = [0, 1]$

is <u>not</u> countable.