3.1 Sequences

A sequence X is a function from $\mathbb{N}$ to $\mathbb{R}$. Sometimes X is defined by a formula for the n-th term x_n such as

$$x_n = \frac{2^n}{n+1}.$$

Sometimes we just define the first few terms,

$$X = \left(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots \right)$$ or

$$x_n = \frac{1}{2n+1}$$

We can also give a recursive formula for x_n:

$$x_n = \frac{x_{n-1}}{x_{n-1}^2 + 1}, \quad x_1 = 3.$$

It is very important to compute the limit of a sequence.

Definition. We say a sequence X converges to x if for all $\varepsilon > 0$, there is a number K in $\mathbb{N}$, so that if $n \geq K$, then $|x_n - x| < \varepsilon$.

The number x is the limit of X, and we say X is convergent.

If X is not convergent, we say X is divergent.

A sequence can only have at most one limit. Suppose $\lim X = x'$ and $\lim X = x''$. Set $\varepsilon = \frac{|x' - x''|}{2}$.

Choose K_1 so $|x_n - x'| < \varepsilon$ if $n \geq K_1$

and choose K_2 so that

$$|x_n - x''| \quad \text{if} \quad n \geq K_2.$$

Now set K = maximum of $\{K_1, K_2\}$.

Then if $n \geq K$,

$$|x' - x''| = |(x' - x_n) - (x'' - x_n)|$$

$$\leq |x' - x_n| + |x'' - x_n|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

$$= |x' - x''|.$$

Dividing by $|x' - x''|$ we get $1 < 1$.

The contraction implies that

$x' = x''$.

Some examples:

Compute $\lim \frac{1}{n}$.

We proved that for any $\varepsilon > 0$, there is a K so that if $n \geq K$, $\frac{1}{n} < \varepsilon$. We obtain that $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon$. It follows that $\lim\left(\frac{1}{n}\right) = 0$.

Ex. Prove that $\lim\left(\frac{3}{n+5}\right) = 0$.

Note that $\frac{3}{n+5} < \frac{3}{n}$.

For a given $\varepsilon > 0$, choose $K > 0$

so that if $n \geq K$, then $\frac{1}{n} < \frac{\varepsilon}{3}$.

If $n \geq K$, then

$$\left| \frac{3}{n+5} - 0 \right| = \frac{3}{n+5} < \frac{3}{n} < 3 \cdot \frac{\varepsilon}{3}$$

$$= \varepsilon.$$

Hence $\lim\left(\frac{3}{n+5}\right) = 0$.

Ex. Show that $\lim (-1)^n$ does not exist.

Assuming $\lim (-1)^n = x$, set $\varepsilon = 1$. Then there is a $K \in \mathbb{N}$ so that if $n \geq K$, then $|(-1)^n - x| < 1$.

If n is even and $\geq K$, then

$$|x - 1| < 1 \rightarrow x - 1 > -1 \rightarrow x > 0$$

If n is odd and $\geq K$, then

$$|x+1| = |x-(-1)^n| < 1.$$

Hence, $x+1 < 1$, which implies that $x < 0$.

This contradiction implies that $\lim (-1)^n$ does not exist.

We now prove

Let (x_n) be a sequence of numbers and let $x \in \mathbb{R}$.
If (a_n) is a sequence of positive numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$, we have

$$|x_n - x| \leq C a_n \quad \text{for all } n \geq m,$$

then it follows that $\lim x_n = x$.

Proof. If $\varepsilon > 0$ is given, then since $\lim(a_n) = 0$, we know there exists K such that $n \geq K$ implies $a_n = |a_n - 0| < \varepsilon/C$.

It follows that if both $n \geq K$ and $n \geq m$, then

$$|x_n - x| \leq C a_n < C(\varepsilon/C) = \varepsilon.$$

Since ε is arbitrary, we conclude that $x = \lim(x_n)$.

We will use this to show that

if $0 < b < 1$, then $\lim(b^n) = 0$.

But first we prove:

Ex. If $a > 0$, show $\lim\left(\frac{1}{1+na}\right) = 0$

Since $a > 0$, then

$0 < na < 1 + na$, and

therefore $0 < \frac{1}{1+na} < \frac{1}{na}$.

Thus we have

$$\left| \frac{1}{1+na} - 0 \right| \le \frac{1}{a} \cdot \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim\left(\frac{1}{n}\right) = 0$,

the above theorem with $C = \frac{1}{a}$

and $m = 1$ implies that

$$\lim\left(\frac{1}{1+na}\right) = 0.$$

Recall that Bernoulli's Inequality states that

if $x > -1$, then

$$(1+x)^n \geq 1+nx, \quad \text{all } n \in \mathbb{N}.$$

We now show that if

$0 < b < 1$, then $\lim(b^n) = 0$.

Since $0 < b < 1$, we can write

$$b = \frac{1}{1+a}$$

where $a := \left(\frac{1}{b}\right) - 1$, so that $a > 0$. By Bernoulli's Inequality, we have

$$(1+a)^n \geq 1 + na, \text{ where } a > -1.$$

Hence,

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na}$$

From the above theorem, we conclude that $\lim(b^n) = 0$.

3.2. Limit Theorems.

Using the results of this section, we can analyse the convergence of many sequences.

Definition. A sequence $X = (x_n)$ is **bounded** if there exists a number $M > 0$ such that

$$|x_n| \leq M, \quad \text{for all } n \in \mathbb{N}.$$

Thm. A convergent sequence of real numbers is bounded.

Pf. Suppose that $\lim x_n = x$ and let $\varepsilon = 1$. Then there is a $K \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq K$. The Triangle In equality with $n \geq K$ implies that

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

If we set

$$M = \max\{|x_1|, |x_2|, \dots |x_{K-1}|, 1+|x|\}$$

then it follows that

$$|x_n| \le M, \quad \text{for all } n \in \mathbb{N}.$$

We want to learn how taking limits interacts with the operations of addition, subtraction, multiplication and division.

Given two sequences $X = (x_n)$ and $Y = (y_n)$, we define

$$X + Y = (x_n + y_n)$$

$$X - Y = (x_n - y_n)$$

$$XY = (x_n y_n)$$

$$cX = (cx_n)$$

and

$$X/Y = \left(\frac{x_n}{y_n}\right) \quad \left(\begin{matrix}\text{providing} \\ y_n \neq 0\end{matrix}\right)$$

Suppose $X = (x_n)$ and $Y = (y_n)$

converge to x and y

respectively. Let $\varepsilon > 0$.

Addition.

Choose K_1 and K_2 so that

$|x_n - x| < \frac{\varepsilon}{2}$ if $n \geq K_1$ and

$|y_n - y| < \frac{\varepsilon}{2}$ if $n \geq K_2$.

Now set $K = \text{Max}\{K_1, K_2\}$

If $n \geq K$, then $n \geq K_1$ and $n \geq K_2$. Hence,

$$|(x_n + y_n) - (x + y)|$$

$$= |(x_n - x) + (y_n - y)|$$

$$\leq |x_n - x| + |y_n - y|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim(x_n + y_n) = x + y$.

For subtraction, we use the same argument. Just replace $x_n + y_n$ by $x_n - y_n$ and $x + y$ by $x - y$.

Multiplication. This is a bit more complicated. Note that

$$|x_n y_n - xy| = \left| (x_n y_n - x_n y) + (x_n y - xy) \right|$$

$$\leq |x_n(y_n - y)| + |(x_n - x)y|$$

$$\leq |x_n||y_n - y| + |x_n - x||y|$$

By the boundedness theorem, there is $M_1 > 0$ such that

$$|x_n| \leq M_1 \quad , \quad \text{all } n.$$

Now set $M = \text{Max}\{M_1, |y|\}$.

We conclude that

$$|x_n y_n - xy| \leq M|y_n - y| + M|x_n - x|$$

Now let $\varepsilon > 0$ be given.

Then there exists K_1

such that

$$|x_n - x| < \frac{\varepsilon}{2M} \quad \text{if } n \geq K_1.$$

Similarly, there exists K_2

such that

$$|y_n - y| < \frac{\varepsilon}{2M} \quad \text{if } n \geq K_2.$$

Now set $K = \text{Max}\{K_1, K_2\}$

If $n \geq K$, then

$|x_n y_n - xy|$

$\leq M|y_n - y| + M|x_n - x|$

$< M \cdot \dfrac{\varepsilon}{2M} + M \cdot \dfrac{\varepsilon}{2M} = \varepsilon.$

This proves

$\lim (x_n y_n) = xy.$