7.2 Trigonometric Integrals

We want to integrate

$$\int \sin^m x \cos^n x \, dx .$$

Ex. Compute $\int \sin^3 x \cos^2 x \, dx$

(Note power of $\sin x$ is odd.)

$$= \int \sin^2 x \sin x \cos^2 x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Let $u = \cos x$ $\quad du = -\sin x \, dx$

$\rightarrow \sin x \, dx = -du$

$$\int (1-u^2)\, u^2 \, (-du)$$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Ex. Compute $\int \sin^2 x \cos^5 x \, dx$

(Power of $\cos x$ is odd.

We can use the above method:)

$$= \int \sin^2 x \cos^4 x \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

$$= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x \, dx$$

Let $u = \sin x \qquad du = \cos x \, dx$

$$\int = \int \left(u^2 - 2u^4 + u^6\right) du$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

When m and n are both even,

it is useful to use

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Ex. Compute $\int_0^{\frac{\pi}{6}} \sin^4 x \, dx$.

First we compute $\int \underline{\sin^4 x \, dx}$

$= \int (\sin^2 x)^2 dx$

$$= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$$

$$= \int \left(\frac{1}{4} - \frac{2\cos 2x}{4} + \frac{\cos^2 2x}{4} \right) dx$$

$$= \int \left(\frac{1}{4} - \frac{\cos 2x}{2} + \frac{(1 + \cos 4x)}{4 \cdot 2} \right) dx$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^4 x \, dx$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \Bigg|_0^{\frac{\pi}{6}}$$

$$= \frac{3\pi}{8 \cdot 6} - \frac{\sin \pi/3}{4} + \frac{\sin \frac{2\pi}{3}}{32}$$

$$= \frac{\pi}{16} - \frac{\sqrt{3}}{2 \cdot 4} + \frac{\sqrt{3}}{2 \cdot 32}$$

$$= \frac{\pi}{16} - \frac{7\sqrt{3}}{64}$$

Ex. Compute $\int \sin^2 x \cos^2 x \, dx$

$$= \int (\sin x \cos x)^2 \, dx$$

$$= \int \left(\frac{\sin 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int \frac{(1 - \cos 4x)}{2} \, dx$$

$$\int \cos ax \, dx = \frac{\sin ax}{a}$$

$$= \frac{x}{8} + \frac{1}{8} \left(\frac{-\sin 4x}{4} \right) + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

$\int \sec^m x \tan^n x \, dx$ is not so hard if m is even or n is odd.

First some special cases:

Ex. Find $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$\therefore \int \tan x \, dx = \ln|\sec x| + C$$

Ex. Compute $\int \sec x \, dx$

$$= \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\sec x + \tan x} \, dx$$

Set $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$\int \; = \int \frac{du}{u} = \ln|u|$$

$$= \ln|\sec x + \tan x| + C$$

$$\therefore \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Note $n = 4$ is even.

Ex. Find $\int \sec^4 x \cdot \tan^2 x \, dx$

$$= \int \sec^2 x \, \sec^2 x \, \tan^2 x \, dx$$

convert to powers of $\tan x$

Now use the identity

$$\sec^2 x = 1 + \tan^2 x$$

$$\int = \int (1+\tan^2 x)\tan^2 x \sec^2 x \, dx$$

Set $u = \tan x \qquad du = \sec^2 x \, dx$

$$\int = \int (u^2 + u^4) \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

For $\int \tan^m x \sec^n x \, dx$,

if n is even and ≥ 2

this method will work:

Factor out $\sec^2 x$, convert everything else to various powers of $\tan x$.

Now suppose m = power of $\tan x$ is odd

Ex. Find $\int \tan^3 x \sec^3 x \, dx$

$$= \int \tan^2 x \sec^2 x \sec x \tan x \, dx$$

Now use $\tan^2 x = \sec^2 x - 1$

$$= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx$$

Set $u = \sec x$ $du = \sec x \tan x \, dx$

$$= \int (\sec^4 x - \sec^2 x) \sec x \tan x \, dx$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

What if $m = 0$ and n is odd?

$\int \sec^3 x \, dx$ is hard

(sec power is odd,

tan power $= 0$ is even)

We integrate by parts!

$u = \sec x \qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad v = \tan x$

Compute $\int \sec^3 x \, dx$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x (\sec x \tan x) \, dx$$

$u = \sec x \quad dv = \sec^2 x$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\therefore 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Reduction Formulas

Compute $\int \cos^n x \, dx$

$$= \int \cos^{n-1} x \cdot \cos x \, dx$$

$u = \cos^{n-1} x \qquad dv = \cos x \, dx$

$du = (n-1) \cos^{n-2} x \, (-\sin x) \, dx$

$v = \sin x$

$$= \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x \, (-\sin^2 x) \, dx$$

$\sin^2 x = 1 - \cos^2 x$

$$= \cos^{n-1}x \sin x + \int (n-1)\cos^{n-2}x \,(1-\cos^2 x)\, dx$$

$$= \cos^{n-1}x \sin x + \int (n-1)\cos^{n-2}x \, dx$$

$$- \int (n-1)\cos^{n}x \, dx$$

$$\therefore\ n \int \cos^n x \, dx = \cos^{n-1}x \sin x$$

$$+ (n-1)\int \cos^{n-2}x \, dx$$

$$\Rightarrow \int \cos^n x \, dx = \frac{1}{n}\cos^{n-1}x \sin x$$

$$+ \frac{n-1}{n}\int \cos^{n-2}x \, dx$$

This is called a Reduction Formula

Use this formula to compute

$$\int_0^{\frac{\pi}{2}} \cos^n x = a_n$$

By the formula:

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}}$$

$$+ \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx$$

$$= 0 + \frac{n-1}{n} a_{n-2}$$