7.3 Trigonometric Substitutions

We use the identities

1. $1 - \sin^2\theta = \cos^2\theta$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $\sec^2\theta - 1 = \tan^2\theta$

to compute integrals like:

$$\int \frac{dx}{\sqrt{1-x^2}} \quad \text{or} \quad \int \frac{dx}{\sqrt{1+16x^2}}$$

1. For $\sqrt{1-x^2}$, let $x = \sin\theta$.

Then $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta} = \cos\theta$

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2. For $\sqrt{1+x^2}$, let $x = \tan\theta$

Then $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta}$

$= \sqrt{\sec^2\theta} = \sec\theta$

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3. For $\sqrt{x^2-1}$, let $x = \sec\theta$

Then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\tan^2\theta} = \tan\theta$

We want to eliminate

$\sqrt{}$. Note that

if we set $x = \sin\theta$,

then $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

Using Id. 1

$= \sqrt{\cos^2\theta} = \cos\theta$

If $x = \sin\theta$, then $dx = \cos\theta\, d\theta$

Ex. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cos\theta\, d\theta}{\cos\theta}$

$= \displaystyle\int d\theta = \theta$

$$x = \sin\theta \rightarrow \sin^{-1}x = \theta$$

$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = \underline{\underline{\sin^{-1}x + C}}$$

Ex. Compute $\displaystyle\int \frac{dx}{\sqrt{1+16x^2}}$

Id. 2. $\Rightarrow$ $1 + \tan^2\theta = \sec^2\theta$

We need $16x^2 = \tan^2\theta$

or $4x = \tan\theta$

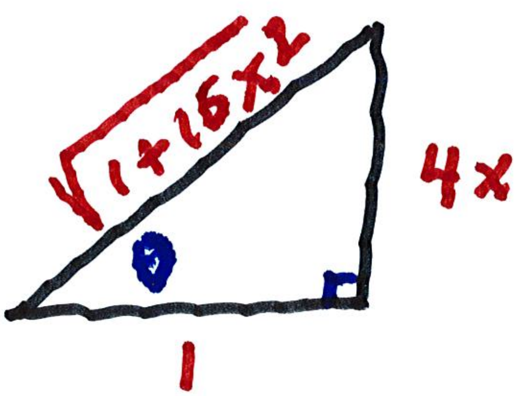
or $x = \frac{1}{4}\tan\theta$

$$\therefore\ dx = \frac{1}{4}\sec^2\theta\ d\theta$$

and $$\int \frac{dx}{\sqrt{1+16x^2}} = \int \frac{\frac{1}{4}\sec^2\theta\ d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{4}\int \sec\theta\ d\theta$$

$$= \frac{1}{4}\ln\left|\sec\theta + \tan\theta\right| + C$$



$\tan\theta = 4x$

$\therefore\ \sec\theta = \sqrt{1+16x^2}$

$$\therefore \frac{1}{4} \ln \left| \sec\theta + \tan\theta \right| + C$$

$$= \frac{1}{4} \ln \left| \sqrt{1+16x^2} + 4x \right| + C$$

Compute $\displaystyle\int \frac{dx}{\sqrt{9x^2-16}}$

Use Identity 3.

$$\sec^2\theta - 1 = \tan^2\theta$$

We want $9x^2 = 16\sec^2\theta$

or $3x = 4\sec\theta$

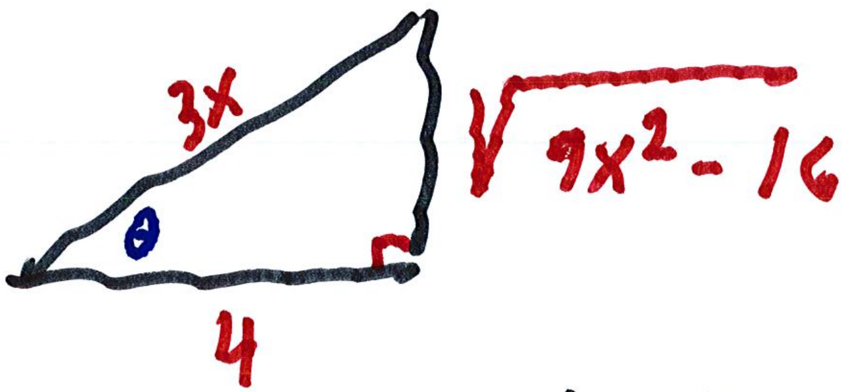
or $x = \frac{4}{3}\sec\theta \rightarrow dx = \frac{4}{3}\sec\theta\tan\theta\, d\theta$

$$\int = \int \frac{\frac{4}{3}\sec\theta\tan\theta\, d\theta}{\sqrt{16\sec^2\theta - 16}} = \int \frac{\sec\theta\tan\theta}{3\tan\theta}\, d\theta$$

$$\int \frac{\sec\theta\, d\theta}{3} = \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

Now convert back to x

$$x = \frac{4}{3}\sec\theta \rightarrow \sec\theta = \frac{3x}{4}$$



$$\therefore \tan\theta = \frac{\sqrt{9x^2-16}}{4}$$

$$\rightarrow \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{3x}{4} + \frac{\sqrt{9x-16}}{4}\right| + C$$

Compute $\displaystyle\int \frac{dx}{x\sqrt{4-9x^2}}$

We use Id. 1. $1-\sin^2\theta = \cos^2\theta$

We want $9x^2 = 4\sin^2\theta$

$\rightarrow 3x = 2\sin\theta \rightarrow 3\,dx = 2\cos\theta\, d\theta$

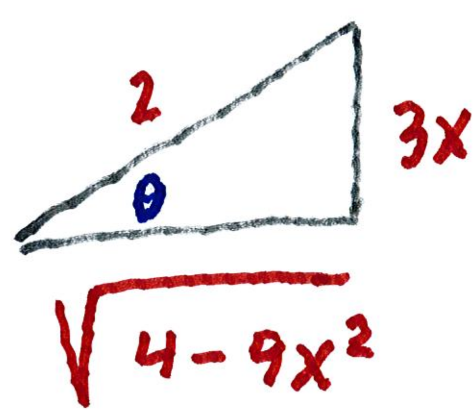
$$\int = \int \frac{\frac{2\cos\theta}{3}\, d\theta}{\frac{2\sin\theta}{3}\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{\cos\theta \, d\theta}{\sin\theta \cdot 2\cos\theta}$$

$$= \frac{1}{2} \int \frac{d\theta}{\sin\theta} = \frac{1}{2} \int \csc\theta \, d\theta$$

$$= \frac{1}{2} \ln \left| \csc\theta - \cot\theta \right| + C$$

Recall $x = \dfrac{2\sin\theta}{3} \rightarrow \sin\theta = \dfrac{3x}{2}$



$$\therefore \quad \csc\theta = \frac{2}{3x} \qquad \cot\theta = \frac{\sqrt{4-9x^2}}{3x}$$

$$\therefore \int = \frac{1}{2} \ln \left| \frac{2}{3x} - \frac{\sqrt{4-9x^2}}{3x} \right| + C$$

Ex. Compute $\int \sqrt{4+25x^2}\,dx$

Id. 2 $\quad 1+\tan^2\theta = \sec^2\theta$

We want $25x^2 = 4\tan^2\theta$

$\rightarrow 5x = 2\tan\theta$

$\rightarrow 5\,dx = 2\sec^2\theta\,d\theta$

$$= \int \sqrt{4+4\tan^2\theta} \cdot \frac{2}{5}\sec^2\theta\,d\theta$$

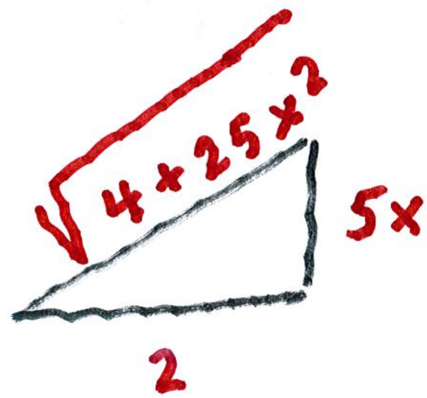
$$= \frac{4}{5}\int \sqrt{1+\tan^2\theta} \cdot \sec^2\theta\,d\theta$$

$$= \frac{4}{5}\int \sec^3\theta \, d\theta$$

$$= \frac{4}{5}\,\frac{1}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right] + C$$

Recall $5x = 2\tan\theta$

$$\rightarrow \frac{5x}{2} = \tan\theta$$



$$\therefore \sec\theta = \frac{\sqrt{4+25x^2}}{2}$$

$$\int = \frac{2}{5}\left[\frac{\sqrt{4+25x^2}}{2}\cdot\frac{5x}{2} + \ln\left|\frac{\sqrt{4+25x^2}}{2} + \frac{5x}{2}\right|\right] + C$$

Ex. Compute $\displaystyle\int \frac{\sqrt{4x^2-1}}{x^2}\,dx$

Use $\sec^2\theta - 1 = \tan^2\theta$

We want $4x^2 = \sec^2\theta$

$\rightarrow 2x = \sec\theta \rightarrow dx = \dfrac{\sec\theta\tan\theta}{2}$

$d\theta$

$$\rightarrow \int \frac{\sqrt{\sec^2\theta - 1}}{\frac{\sec^2\theta}{4}} \cdot \frac{\sec\theta\tan\theta}{2}\,d\theta$$

$$= \int \frac{2\tan^2\theta \cdot \sec\theta}{\sec^2\theta}\, d\theta$$

$$= 2\int \frac{\tan^2\theta \, d\theta}{\sec\theta}$$

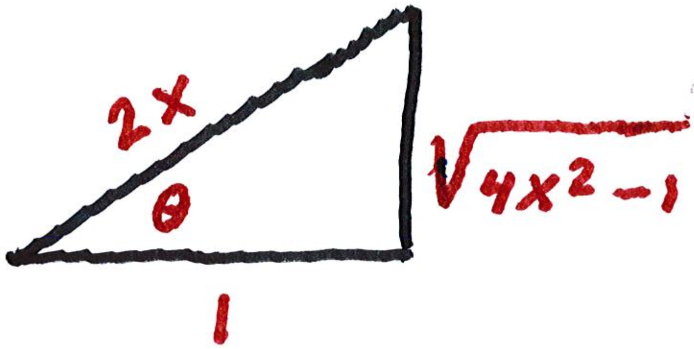
$\tan^2\theta = \sec^2\theta - 1$

$\sec^2\theta = 1 + \tan^2\theta$

$$= 2\int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta$$

$$= 2\int \sec\theta \, d\theta - 2\int \cos\theta \, d\theta$$

$$= 2\ln\left|\sec\theta + \tan\theta\right| - 2\sin\theta + C$$



$$\sec\theta = 2x = \frac{2x}{1}$$

$$\therefore \tan\theta = \sqrt{4x^2-1} \qquad \sec\theta = 2x$$

$$\sin\theta = \frac{\sqrt{4x^2-1}}{2x}$$

$$\therefore \int = 2\ln\left|2x + \sqrt{4x^2-1}\right| - \frac{2\sqrt{4x^2-1}}{2x} + C$$