7.4 cont'd.

Ex 1: Compute $\int \frac{x^3 - x + 2}{x^2 - 3x - 4}\, dx$

1. Do long division:

$$
\begin{array}{r}
x + 3 \\
x^2 - 3x - 4 \,\overline{\big)\, x^3 + 0x^2 - x + 2} \\
\underline{x^3 - 3x^2 - 4x} \\
3x^2 + 3x + 2 \\
\underline{3x^2 - 9x - 12} \\
12x + 14
\end{array}
$$

$\therefore \frac{x^3 - x + 2}{x^2 - 3x - 4} = x + 3 + \frac{12x + 14}{x^2 - 3x - 4}$

2. Factor denominator

into linear or quadratic factors

$$\frac{12x+14}{x^2-3x-4} = \frac{12x+14}{(x-4)(x+1)}$$

3. Decompose into partial

fractions.

$$\frac{12x+14}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

Write all terms with same

common denominator.

$$\frac{A}{x-4} + \frac{B}{x+1} = \frac{A(x+1)}{(x-4)(x+1)}$$

$$+ \frac{B(x-4)}{(x+1)(x-4)}$$

4. Compare numerators

$$A(x+1) + B(x-4) = 12x + 14$$

$$\rightarrow \quad A + B = 12 \quad \rightarrow 5B = -2$$

$$A - 4B = 14 \qquad B = -\frac{2}{5} \leftarrow$$

$$A = 12 + 2/5$$

$$= 62/5 \leftarrow$$

Integrate

$$\int \frac{62}{5}\frac{1}{x-4} - \frac{2}{5}\cdot\frac{1}{x+1}\,dx$$

$$\int \frac{x^3 - x + 2}{x^2 - 3x - 4}\,dx$$

$$= \left(\frac{x^2}{2} + 3x\right) + 62/5 \ln|x-4|$$

$$- \frac{2}{5}\ln|x+1| + K$$

Ex. Compute $\displaystyle\int_2^3 \frac{x^2+2}{x^3 - x}\,dx$

First we compute the

<u>indefinite</u> integral

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{x^2+2}{x^3-x}$$

Use common denominator:

$$A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = x^2+2$$

x^2-coef. $\quad A + B + C = 1$

x-coef. $\quad 0 + B - C = 0$

1-coef. $\quad -A + 0 + 0 = 2$

$\therefore \underline{A = -2}$

$$\left.\begin{aligned} B + C &= 3 \\ B - C &= 0 \end{aligned}\right\} \underline{B = \tfrac{3}{2}}$$

$$\rightarrow \underline{C = \tfrac{3}{2}}$$

$$R(x) = \frac{-2}{x} + \frac{3}{2}\cdot\frac{1}{x-1} + \frac{3}{2}\cdot\frac{1}{x+1}$$

$$\int R(x)\,dx = -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{3}{2}\ln|x+1|$$

$$= \ln\left|\frac{(x-1)^{3/2}(x+1)^{\frac{3}{2}}}{x^2}\right| + K$$

$$\int_2^3 R(x) = \ln\left(\frac{2^{3/2}\cdot 4^{3/2}}{3^2}\right) - \ln\left(\frac{3^{\frac{3}{2}}}{2^2}\right)$$

Now suppose the denominator has some repeated factors.

Ex. Compute $\displaystyle\int \frac{2x^2 - x}{(x-1)(x+1)^2}\,dx$ 7, 8 not here

Decompose:

$$\frac{2x^2 - x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$2x^2 - x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$2x^2 - x = A(x^2 + 2x + 1)$$
$$+ B(x^2 \quad -1)$$
$$+ C(\quad x - 1)$$

Compare Coefficient

x^2: $2 = A + B$

x $-1 = 2A + C$

1: $0 = A - B - C$

Add E1 and E_3

$\hookrightarrow$ $2A - C = 2$

E_2 $2A + C = -1$ $\rightarrow$

$4A = 1$

$$\rightarrow A = \frac{1}{4}$$

E_2 $\therefore -1 = \frac{1}{2} + C \rightarrow C = -\frac{3}{2}$

E_1 $2 = \frac{1}{4} + B \rightarrow B = \frac{7}{4}$

$$\int = \int \frac{1}{4}\cdot\frac{1}{x-1} + \frac{7}{4}\cdot\frac{1}{x+1} - \frac{3}{2}\cdot\frac{1}{(x+1)^2} \, dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{7}{4}\ln|x+1| + \frac{3}{2}\cdot\frac{1}{x+1} + K$$

$$= \cancel{\ln|x| - \frac{2}{(x-1)} + K}$$

Ex. Find $\displaystyle\int \frac{(x^2+2)}{(x-1)^3(x+2)}\,dx$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+2}$$

$$\Rightarrow A(x-1)^2(x+2) + B(x-1)(x+2)$$

$$+ C(x+2) + D(x-1)^3 = x^2+2$$

x^3- coef. $= A \qquad + D = 0$

x^2- coef. $\qquad B \quad - 3D = 1$

x- coef. $\; -3A + B + C + 3D = 0$

1- coef. $\quad 2A - 2B + 2C - D = 2$

$\rightarrow \; A = \frac{2}{9}, \quad B = \frac{1}{3}, \quad C = 1, \quad D = -\frac{2}{9}$

$$\therefore R(x) = \frac{2}{9}\frac{1}{(x-1)} + \frac{1}{3}\frac{1}{(x-1)^2}$$

$$+ \frac{1}{(x-1)^3} - \frac{2}{9}\frac{1}{x+2}$$

$$\rightarrow \int R(x)\,dx = \ln|x-1|^{2/9} - \ln|x+2|^{2/9}$$

$$- \frac{1}{3}\frac{1}{(x-1)} - \frac{1}{2(x-1)^2} + K$$

In general:

$$\int \frac{2x^3 - 1}{(x-1)(x+2)^3(x+4)^2}\,dx$$

$$= \int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$+ \frac{E}{x+4} + \frac{F}{(x+4)^2}\,dx$$

Use common denominators:

and compare numerators

Then look at coefficients

of x^k for $k = 0, 1, 2, \dots, 5$

Yields 6 equations in

6 variables.

Ex: Compute $\displaystyle\int \frac{(x^2+1)\,dx}{(x+1)(x^2+4)}$

x^2+4 is an irreducible

x^2+4

quadratic

$= (x+2i)(x-2i)$

We write

$$\frac{x^2-4}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{(Bx+C)}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$$

x^2: $\quad A + B = 1$

x: $\quad -B + C = 0$

1: $\quad 4A - C = -4$

Then $A = -\frac{3}{5}$, $B = \frac{8}{5}$ $\quad C = \frac{8}{5}$

$$\int \frac{x^2 - 2}{(x-1)(x^2+4)}\, dx$$

$$= \int \frac{-\frac{3}{5}}{x-1}\, dx + \int \frac{\frac{8}{5}x + \frac{8}{5}}{x^2+4}\, dx$$

$$= -\frac{3}{5}\ln|x-1| + \frac{8}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$$

$$+ \int \frac{\frac{4x}{5}}{x^2+4}\, 2\, dx$$

$$= \frac{4}{5}\int \frac{2x}{x^2+4}\, dx$$

$$= \frac{3}{5} \ln (x^2 + 4)$$

So, $\displaystyle\int \frac{x^2 - 2}{(x-1)(x^2+4)}\, dx$

$$= -\frac{3}{5} \ln |x-1| + \frac{4}{5} \tan^{-1}\left(\frac{x}{2}\right)$$

$$+ \frac{4}{5} \ln (x^2 + 4) + K$$

Consider

$$\int \frac{2x^{12} + 2}{(x-4)^2 (x+1)^3 (x^2+9)^3 (x^2+3)} \, dx$$

$$= \int \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

$$+ \frac{Fx + G}{x^2+9} + \frac{Hx + I}{(x^2+9)^2} + \frac{Jx + K}{(x^2+9)^3}$$

$$+ \frac{Lx + M}{(x^2+3)} \, dx$$

To compute $\int \frac{dx}{(x^2+a^2)^k}$, for

$k = 1, 2, \ldots,$ one uses a

reduction formula.