11.1 Sequences

A sequence is a list of numbers written in a definite order.

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We sometimes write

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Sometimes we give a formula for the n-th term.

Sometimes we list the first several terms

$\left\{ 2^{-n} \right\}_{n=1}^{\infty}$ $a_n = 2^{-n}$ $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \right\}$

$\left\{ \frac{n}{n-2} \right\}_{n=3}^{\infty}$ $a_n = \frac{n}{n-2}$

$$\left\{ \frac{3}{1}, \frac{4}{2}, \frac{5}{3}, \dots, \frac{n}{n-2}, \dots \right\}$$

Sometimes it's hard to find a formula:

$$\left\{ -\frac{2}{2}, \frac{4}{3}, -\frac{8}{4}, \frac{16}{5}, \dots \right\}$$

$2, 4, 8, 16$ $\quad 2^n$

$2, 3, 4, 5$ $\quad n+1$

$$a_n = \frac{2^n}{n+1}$$ Need - signs:

$$a_n = \frac{(-1)^n 2^n}{n+1}$$

Sometimes $\{a_n\}$ is defined recursively:

Set $a_1 = 1$ and $a_n = 2a_{n-1} + n^2$ when $n \geq 2$.

$$\{1, 6, 21, 58, 141, \ldots\}$$

Finding a formula $f(n)$ for the n-th term seems very hard

We want to define the limit of a sequence.

We want to say that

$$\lim_{n \to \infty} a_n = L$$

if the n-th term gets closer and closer to L as n increases.

But if $a_n = 1 + (-1)^n$ and $L = 2$ then $a_n = 2$ if n is even.

Is this good enough?

Precise Definition of $\lim_{n\to\infty} a_n$.

We say $\lim_{n\to\infty} a_n = L$ if

for every number $\varepsilon > 0$

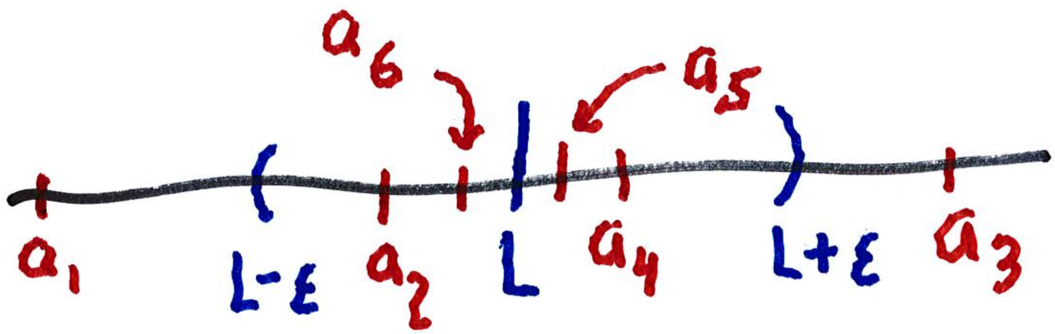
there is an integer $N_\varepsilon > 0$

such that if $n > N_\varepsilon$, then

$|a_n - L| < \varepsilon$. In this case,

we say $\{a_n\}$ is convergent

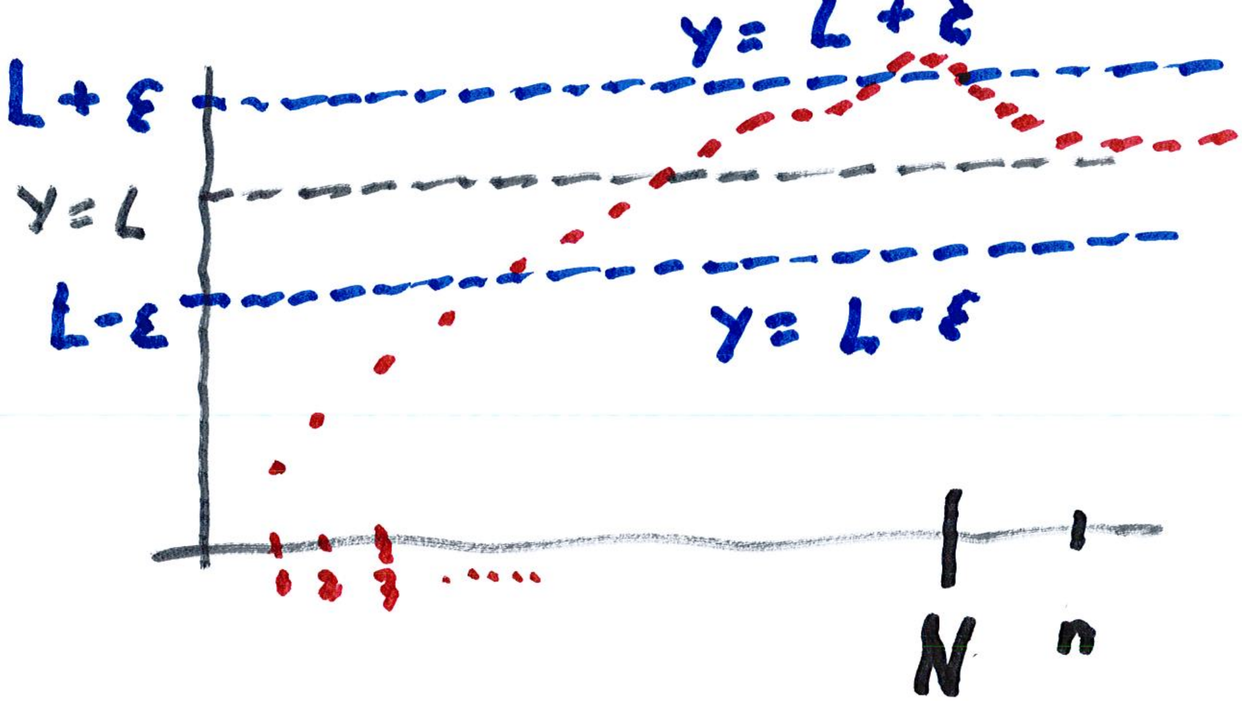
In other words, if n is large enough, say $n > N_\varepsilon$, then a_n is close to L, i.e. $|a_n - L| < \varepsilon$



In this case it seems that $N_\varepsilon = 3$ works.

$|a_n - L| < \varepsilon$

$-\varepsilon < a_n - L < \varepsilon \rightarrow \quad L-\varepsilon < a_n < L+\varepsilon$



If $n > N$, then $|a_n - L| < \varepsilon$

Thm: If $\lim_{x \to \infty} f(x) = L$,

and $a_n = f(n)$, then

$$\lim_{n \to \infty} a_n = L$$

Def'n: $\lim_{n \to \infty} a_n = \infty$ means

that for every positive

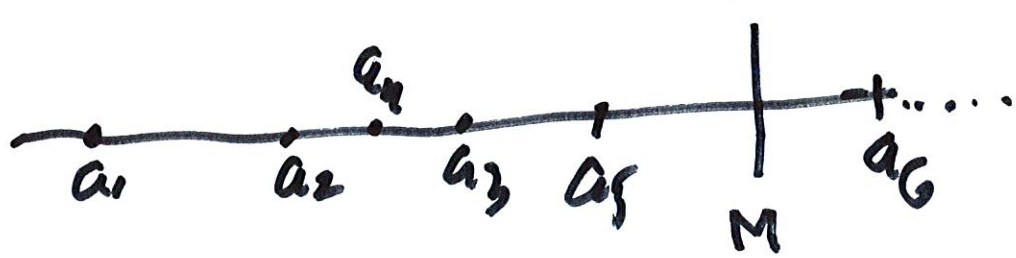
number M, there is an

integer N so that

if $n > N$ then $a_n > M$

Such a sequence is divergent.

but still in a useful way.



Limit Laws

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences. Then

1. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim a_n + \lim b_n$

2. $\lim\limits_{n\to\infty} (a_n - b_n) = \lim a_n - \lim b_n$

3. $\lim\limits_{n\to\infty} c a_n = c \lim a_n$

4 $\lim\limits_{n\to\infty} (a_n b_n) = \lim a_n \cdot \lim b_n$

5. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim a_n}{\lim b_n}$,

(if $\lim b_n \neq 0$)

6. $\lim\limits_{n\to\infty} a_n^p = \left\{\lim a_n\right\}^p$ if $p > 0$ and $a_n > 0$.

Ex. Show $\lim\limits_{n\to\infty} \dfrac{2n+1}{3n+2} = \dfrac{2}{3}$

$$\frac{2n+1}{3n+2} = \frac{n\left(2+\frac{1}{n}\right)}{n\left(3+\frac{2}{n}\right)} = \frac{2+\frac{1}{n}}{3+\frac{2}{n}}$$

For functions, we know that

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \quad \text{if } r > 0$$

$$\therefore \lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

(by 6)

$$\therefore \lim_{n \to \infty} \frac{1}{n} = 0 \quad (\text{if } r = 1)$$

$$\therefore \lim_{n \to \infty} 2 + \frac{1}{n} = 2 + 0 = 2$$

(by 1)

and $\displaystyle \lim_{n \to \infty} \frac{2}{n} = 2 \cdot 0 = 0$

(by 3)

$$\therefore \lim_{n\to\infty}\left(3+\frac{2}{n}\right) = 3+0 = 3$$
(by 1)

$$\therefore \lim_{n\to\infty}\frac{2+\frac{1}{n}}{3+\frac{2}{n}} = \frac{2}{3}$$ (by 5)

Thm. If $\lim_{n\to\infty} a_n = L > 0$

and $\lim_{n\to\infty} b_n = \infty$, then

$$\lim_{n\to\infty} a_n b_n = \infty$$

Ex. $d_n = \sqrt{4n^2+n} = n\sqrt{4+\frac{1}{n}}$
$\downarrow$ $+\infty$, $\downarrow$ 2

Ex. Show that $\lim_{n\to\infty} \frac{n}{\sqrt{7+2n}} = \infty$

Note that $\frac{n}{\sqrt{2n+7}} = \frac{\sqrt{n}\cdot\sqrt{n}}{\sqrt{2n+7}}$

$$= \underset{\substack{\downarrow \\ \infty}}{\sqrt{n}}\sqrt{\frac{n}{2n+7}} \to \sqrt{\frac{1}{2}} > 0$$

$$= \sqrt{n}\cdot\sqrt{\frac{1}{2+\frac{7}{n}}} \to \infty$$

$$\lim_{n\to\infty} \sqrt{n} = \infty$$

and $\displaystyle\lim_{n\to\infty} \sqrt{\frac{1}{2+\frac{7}{n}}} = \sqrt{\frac{1}{2}} > 0$

$$\therefore \lim_{n\to\infty} \sqrt{n}\cdot\sqrt{\frac{1}{2+\frac{7}{n}}} = \infty$$

Ex. Show $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$

L'Hôpital's Rule $\Rightarrow$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Hence, $\displaystyle\lim \frac{\ln n}{n} = 0$

Ex.

Show $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$

$$\lim_{x\to\infty} \frac{x^2}{2^x} \underset{L'H.}{=} \lim_{x\to\infty} \frac{2x}{(\ln 2)\, 2^x}$$

$$\underset{L}{=} \lim_{x\to\infty} \frac{2}{(\ln 2)^2\, 2^x} = 0$$

$$= \lim_{n\to\infty} \frac{n^2}{2^n} = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

$$\therefore \lim_{n \to \infty} \frac{\ln n}{n} = 0.$$

Squeeze Thm:

Suppose $a_n \le b_n \le c_n$

for all $n \ge n_0$, and that

$$\lim_{n \to \infty} a_n = L \quad \text{and} \quad \lim_{n \to \infty} c_n = L$$

Then $\lim_{n \to \infty} b_n = L$

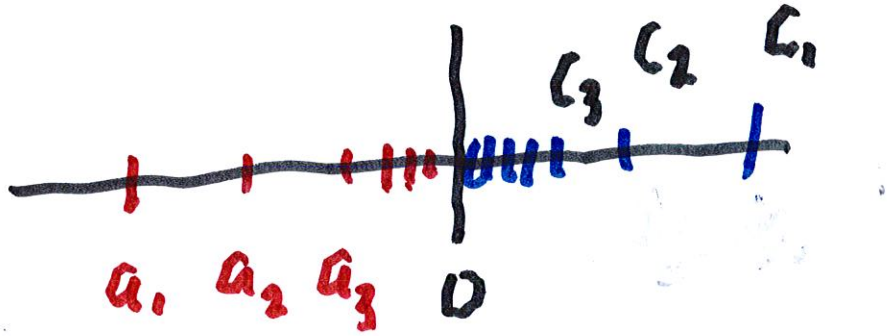
Ex. Let $c_n = \dfrac{n}{3n^2+2}$

$$= \frac{n}{n^2\left(3+\frac{2}{n^2}\right)} = \underbrace{\frac{1}{n}}_{\downarrow\, 0} \cdot \underbrace{\frac{1}{3+\frac{2}{n^2}}}_{\downarrow\, \frac{1}{3}}$$

$$\therefore \lim_{n\to\infty} \frac{n}{3n^2+2} = 0$$

Let $b_n = (-1)^n \cdot \dfrac{n}{3n^2+2}$

and $a_n = -\dfrac{n}{3n^2+2} \to -0 = 0$



$$-\frac{n}{3n^2+2} \le (-1)^n \cdot \frac{n}{3n^2+2} \le \frac{n}{3n^2+2}$$

$$a_n \to 0 \qquad\qquad c_n \to 0$$

$$\therefore \lim_{n\to\infty} \frac{(-1)^n n}{3n^2+2} = 0$$

$$\frac{-\sqrt{n}}{2n+3} \le \frac{(-1)^n\sqrt{n}}{2n+3} \le \frac{\sqrt{n}}{2n+3}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad\qquad 0$$

$$\therefore \lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{2n+3} = 0 \quad \text{too.}$$

Thm. Suppose $\lim_{n\to\infty} a_n = L$

and that f is continuous

at L. Then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Ex. Compute $\lim\limits_{n\to\infty} \tan^{-1}\left(\frac{n+2}{n+4}\right)$

Set $a_n = \frac{n+2}{n+4} = \frac{1+\frac{2}{n}}{1+\frac{4}{n}} \to \frac{1}{1} = 1$

Also $\tan^{-1}x$ is continuous at $x=1$

$$\Rightarrow \lim_{x\to 1} \tan^{-1}x = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore \lim_{n\to\infty} \tan^{-1}(a_n) = \tan^{-1} 1 = \frac{\pi}{4}$$

Two useful limits :

If $r > 1$, then $\lim\limits_{n\to\infty} r^n = \infty$

Ex. $\lim\limits_{n\to\infty} 2^n = \infty$, $\lim\limits_{n\to\infty} (1.1)^n = \infty$

If $0 < r < 1$, then $\lim\limits_{n\to\infty} r^n = 0$

Ex. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$

and $\lim_{n\to\infty} (.99)^n = 0$

If $-1 < r < 0$,

$$\lim_{n\to\infty} r^n = 0$$

Ex. $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$

Def'n. A sequence $\{a_n\}$ is called

increasing if $a_n < a_{n+1}$

for all $n \geq 1$

Similarly, $\{a_n\}$ is called

decreasing if $a_n > a_{n+1}$

for all $n \geq 1$

Ex. Show $a_n = 2 - \frac{3}{n}$ is inc.

(If $\{a_n\}$ is either increasing or decreasing, then $\{a_n\}$ is monotonic)

Set $f(x) = 2 - \frac{3}{x}$

$f'(x) = \frac{3}{x^2} > 0$ if $x > 0$

$\therefore$ if $x_1 < x_2$, then

$f(x_1) < f(x_2)$

Hence, $f(n) < f(n+1)$ if $n \geq 1$

or $2 - \frac{3}{n} < 2 - \frac{3}{n+1}$

$\therefore$ $a_n = 2 - \frac{3}{n}$ is inc.

Def'n. We say $\{a_n\}$ is bounded above if there is a number M so that

$$a_n \leq M, \text{ for all } n.$$

Similarly, $\{a_n\}$ is bounded below if $m \leq a_n$, for all $n \geq 1$

Monotonic Seq. Thm.

If $\{a_n\}$ is inc. and bounded above, then there is a number L so $\lim_{n\to\infty} a_n = L$