11.2 Infinite Series

We can write

$$a_1 + a_2 + \dots + a_N = \sum_{n=1}^{N} a_n$$

When can we give a meaning to $\sum_{n=1}^{\infty} a_n$

$$= a_1 + a_2 + a_3 + \dots + a_n + \dots \quad ?$$

For example:

$\pi = 3.14159\,26535$ etc.

$$= 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10{,}000} + \frac{9}{100{,}000} + \ldots$$

The more terms we add up, the closer the sum is to π.

Clearly, the series

$1 + 2 + 3 + \ldots$ doesn't add

up to a finite number

Ex. What about

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots \quad ?$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{2^2}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{2^3}$$

$$\vdots$$

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Letting $n \to \infty$,

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots = 1$$

Given any sequence $\{a_n\}_{n=1}^{\infty}$

we define

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_n}_{\text{a partial sum.}}$$

We write $\sum_{i=1}^{n} a_i = \underbrace{a_1 + a_2 + \ldots + a_n}_{S_n}$

we call $\sum_{n=1}^{\infty} a_n$ an infinite series.

When does the series have a meaning?

Def'n. Given the series $\sum_{n=1}^{\infty} a_n$, we define

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

S_n is called the n-th partial sum. If the sequence $\{S_n\}$ is convergent and if $\lim_{n\to\infty} S_n = S$, $\left\{\begin{matrix}\text{where} \\ S \text{ is finite}\end{matrix}\right\}$

then we say $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = S$$

$$\text{or } \sum_{n=1}^{\infty} a_n = S.$$

If $\{S_n\}$ is divergent, we say

the series is divergent

Ex. If $\sum_{n=1}^{\infty} a_n$ is a series,

and if $S_n = \dfrac{2n-1}{3n+2}$, for $n = 1, 2, \dots$,

then $\lim_{n \to \infty} \dfrac{2n-1}{3n+2} = \dfrac{2}{3}$

$\therefore \sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = \frac{2}{3}$$

Geometric Series

For the series $\sum_{n=0}^{\infty} r^n$.

define $S_n = 1 + r + r^2 + \cdots + r^n$

Note: $rS_n = r + r^2 + \cdots + r^n + r^{n+1}$

Subtract:

$$S_n(1-r) = 1 - r^{n+1}$$

or $$S_n = \frac{1 - r^{n+1}}{1-r}$$

(if $r \neq 1$)

We know $\lim_{n\to\infty} r^n = 0$

if $\underline{|r| < 1}$

$$\therefore \frac{r}{1-r} \cdot r^n = \frac{r^{n+1}}{1-r} \to 0$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^{n+1}}{1-r}\right)$

$$= \frac{1}{1-r}$$

$$\therefore \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r| < 1$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

We also see that

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r|<1.$$

Note that $\sum_{n=0}^{\infty} ar^n$ is a

geometric series with

first term = a, and

shrinking factor = r

It's sum is $\dfrac{a}{1-r}$

Ex. Consider $\sum_{n=0}^{\infty} \frac{3 \cdot 2^{n+1}}{5^n} = S$

What is S?

$a = a_0 = \frac{3 \cdot 2}{1}$

$= 6$

$r = \frac{a_1}{a_0}$ $\quad a_1 = \frac{3 \cdot 2^2}{5} = \frac{12}{5}$

$\therefore \frac{a_1}{a_0} = \frac{\frac{12}{5}}{6} = \frac{2}{5}$

$\therefore S = \frac{a}{1-r} = \frac{6}{1-\frac{2}{5}} = \frac{6}{\frac{3}{5}} = \underline{\underline{10}}$

Ex. Find $S = \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{3^{n-1}}$

$n=1 \rightarrow a_1 = \frac{(-2)^2}{1} = 4$

$n=2 \rightarrow a_2 = \frac{(-2)^3}{3} = -\frac{8}{3}$

$\therefore r = \frac{a_2}{a_1} = \frac{-\frac{8}{3}}{4} = -\frac{2}{3}$

$\therefore S = \frac{a}{1-r} = \frac{4}{1-(-2/3)} = \frac{4}{\frac{5}{3}}$

$= \frac{12}{5}$

Ex. Evaluate

$$4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots$$

$a = 4$ Shrinking factor is

$$\frac{-\frac{8}{3}}{4} = -\frac{8}{12} = -\frac{2}{3}$$

$\therefore r = -\frac{2}{3}$

So, $S = \dfrac{4}{1-\left(-\frac{2}{3}\right)} = \dfrac{4}{\frac{5}{3}} = \underline{\dfrac{12}{5}}$

Ex. Show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

By partial fractions:

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

$$\rightarrow S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$+ \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

By cancellation.

$$S_n = 1 - \frac{1}{n+1}$$

$$\therefore S_n = 1 - \frac{1}{n+1}$$

for $n = 1, 2, \ldots$

$$\lim_{n \to \infty} S_n = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Ex. Find $S = \sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

$$\frac{1}{n(n+2)} = \frac{a}{n} + \frac{b}{n+2} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}$$

$$1 = a(n+2) + bn = (a+b)n + 2a$$

$$1 = 2a \rightarrow a = \frac{1}{2}$$

and $b = -a = -\frac{1}{2}$

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

Compute $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

$$S_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$+ \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$+ \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\lim S_n = \frac{3}{2} \quad \therefore$$

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \underline{\underline{\frac{3}{2}}}$$

Very Important Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad (\text{harmonic series})$$

$\rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{4}{8} = 1 + \frac{3}{2}$$

$$S_{16} > 1 + \frac{3}{2} + \frac{8}{16} = 1 + \frac{4}{2}, \text{ etc.}$$

In general,

$$S_{2^k} > 1 + \frac{k}{2} \to \infty \quad \text{as } k \to \infty$$

$\therefore \{S_n\}$ does not converge to a finite number In fact

$$\lim_{n\to\infty} S_n = \infty.$$

This example shows that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (and adds up to ∞) EVEN THOUGH $\lim_{n\to\infty} \frac{1}{n} = 0$

But we do have the following:

Thm. If $\sum_{n=1}^{\infty} a_n$ is convergent

(that is, it adds up to a

finite number), then

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Since $\sum_{n=1}^{\infty} a_n$ converges

to S, it follows that

$\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S.$

$$a_n = S_n - S_{n-1} \to S - S = 0$$

as $n \to \infty$

$\therefore \; a_n \to 0$ as $n \to \infty$

Ex. Does $\displaystyle\sum_{n=1}^{\infty} e^{1/n}$ converge?

$\frac{1}{n} \to 0$ as $n \to \infty$

e^x is cont. at 0

$$\therefore \lim_{n\to\infty} e^{1/n} = e^0 = 1 \neq 0$$

$$\therefore \sum_{n=1}^{\infty} e^{1/n} \text{ diverges}$$

Test for divergence:

If $\lim\limits_{n\to\infty} a_n$ does not exist

OR if $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$

diverges

Thm. If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are

convergent series, then

(1) $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$

(2) $$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

(3) $$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Evaluate $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2^n} + \frac{5}{n(n+1)} \right)$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{3}{2^n} = 3$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5$$

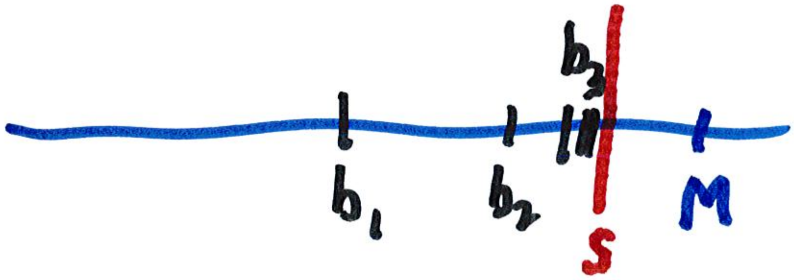
$$\therefore \sum_{n=1}^{\infty} \frac{3}{2^n} + \frac{5}{n(n+1)} = 3 + 5 = 8$$

From 11.1 Suppose $\{b_n\}$ is

bounded and increasing.

Then there is an s so

$$\lim_{n \to \infty} b_n = s$$



Ex. Does $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$ converge.

Note that $\displaystyle\sum_{n=1}^{\infty}\frac{3}{5^n}$ converges

If first series converges, so does

$\displaystyle\sum_{n=1}^{\infty}\frac{2}{n}$ (by subtraction)

If this converges, then so does

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ (mult. by $\frac{1}{2}$)

Does $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ converge

$$\underbrace{\frac{n^2+1}{2n^2+1}}_{c_n}$$

$\lim c_n = \frac{1}{2}$ $\qquad \ln x$ cont. at $x = \frac{1}{2}$

$\therefore \ln(c_n) \to \ln\left(\frac{1}{2}\right)$ as $n \to \infty$

Since $\ln\left(\frac{1}{2}\right) \neq 0$, series diverges

$$\lim \left(\frac{n^2+1}{2n^2+1}\right) = \lim \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(2+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2}$$

Relative Growth Rates

$\ln n \ll n^p \ll c^n \ll n!$

$(p>0) \quad (c>1)$

$\ll n^n$

Ex. Show $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

For large n, $n^3 < (1.1)^n$

$$\Rightarrow \frac{n^3}{2^n} < \frac{(1.1)^n}{2^n} = \left(\frac{1.1}{2}\right)^n$$

(if n is large)