11.5 Alternating Series

An alternating series is a series whose terms are alternately positive and negative

Ex $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}$

Ex. $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$$

In general, $\sum_{n=1}^{\infty} a_n$ is alternating

if $a_n = (-1)^n b_n$ OR $a_n = (-1)^{n-1} b_n$

where $b_n > 0$.

If a series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

or $\sum_{n=1}^{\infty} (-1)^n b_n$ is alternating,

then it converges if it satisfies

(i) $b_{n+1} \le b_n$, for all n and

(ii) $\lim_{n \to \infty} b_n = 0$

This is called the Alternating Series Test

Ex. For the first example

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2n-1},$$

$b_n = \frac{1}{2n-1}$ is decreasing ✓

(the denominator is increasing)

and

$\lim_{n\to\infty} \frac{1}{2n-1} = 0$ ✓ ∴ Series Converges

Ex. For the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$,

$b_n = \frac{n}{n+2}$ satisfies

$$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = \frac{1}{1+0}$$

$$= 1$$

$\therefore$ (ii) is <u>not</u> satisfied.

Hence $\lim_{n\to\infty} (-1)^n \frac{n}{n+2}$

does not exist. Series Diverges

Ex. Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge?

The series is alternating.

Also $\sqrt{n}$ is increasing

$$\left(\sqrt{n+1} > \sqrt{n} \right)$$

$\therefore \frac{1}{\sqrt{n}}$ is decreasing, ✓

and $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ ✓

$\therefore$ Series converges.

Ex. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3}{e^n}$

is alternating.

(i) To show $\left\{ \dfrac{n^3}{e^n} \right\}$ is decreasing,

we can define $f(x) = \dfrac{x^3}{e^x}$.

$$f'(x) = \frac{e^x \cdot 3x^2 - x^3 \cdot e^x}{e^{2x}}$$

$$= \frac{x^2(3 - x)}{e^x} < 0 \quad \text{if } x > 3.$$

This shows f is decreasing on $[3, \infty)$.

In particular $b_{n+1} < b_n$

if $n \geq 3$ $\Rightarrow$ (i) holds.

To show $\lim\limits_{n\to\infty} \frac{n^3}{e^n} = 0$, we

use L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{x^3}{e^x} \underset{L}{=} \lim_{x\to\infty} \frac{3x^2}{e^x}$$

$$\underset{L}{=} \lim_{x\to\infty} \frac{6x}{e^x} \underset{L}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

$\Rightarrow \lim\limits_{n\to\infty} b_n = 0$.

It follows that

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-1} n^3}{e^n} \text{ converges}$$

But then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{e^n}$

also converges

Ex. Find if $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos n\pi}{e^n}$ converges

$$\cos n\pi = (-1)^n$$

$\therefore$ Series is $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n}$

$$f(x) = \frac{x}{e^x} \to f'(x) = \frac{e^x \cdot 1 - xe^x}{e^{2x}}$$

$$= \frac{(1-x)}{e^x} < 0 \text{ if } x > 1$$

$\therefore f$ is decreasing if $x \geq 1$

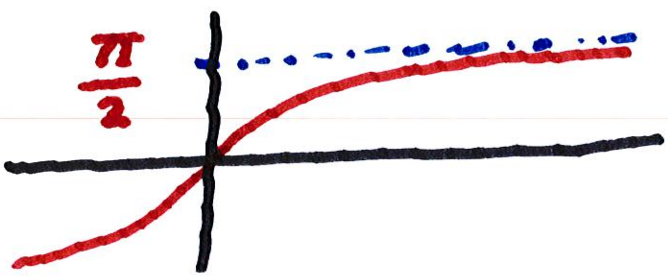
$\Rightarrow b_n = \frac{n}{e^n}$ is dec. ✓

$$\lim_{x \to \infty} \frac{x}{e^x} \underset{\mathcal{L}}{=} \lim_{x \to \infty} \frac{1}{e^x} = 0 \quad ✓$$

$$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n} \text{ converges}$$

Ex. Find if $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$ converges.

$b_n = \arctan n \to \frac{\pi}{2}$ as $n \to \infty$



Since b_n does not $\to 0$,

the series diverges

To explain why the Alt. Ser. Test is true, consider the even partial sums:

$$S_2 = b_1 - b_2 \geq 0 \qquad (b_1 \geq b_2)$$

$$S_4 = S_2 + (b_3 - b_4) \geq S_2$$

$$(b_3 \geq b_4)$$

$$\vdots$$

$$S_{2n} = S_{2n-2} + (b_{2n-1} - b_{2n}) \geq S_{2n-2}$$

$$(b_{2n-1} - b_{2n} \geq 0)$$

$$\therefore\ 0 \le S_2 \le S_4 \le S_6 \le \dots \le S_{2n} \le \dots$$

On the other hand,

$$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5)$$

$$\dots - (b_{2n-2} - b_{2n-1})$$

$$- b_{2n}$$

But $b_2 - b_3 \ge 0$,

and $(b_4 - b_5) \ge 0$,

$\vdots$

$\therefore \quad S_{2n} \le b_1$ for all n.

Monotonic Sequence Thm:

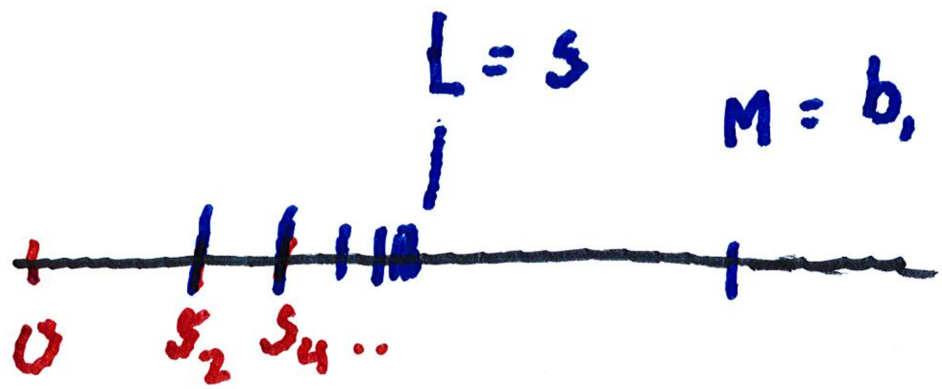
Suppose $\{a_n\}$ is increasing

and $\{a_n\}$ is bounded above

(i.e., $a_n \le M$ for all n).

Then there is a number $L \le M$

so $\lim\limits_{n \to \infty} a_n = L$



$\therefore$ There is a number $S \le b_1$

so $\lim\limits_{n \to \infty} S_{2n} = S.$

But we also have

$$S_{2n+1} = S_{2n} + b_{2n+1}$$

$$\therefore \lim_{n \to \infty} S_{2n+1} = \lim S_{2n} + \lim b_{2n+1}$$

$$= S + 0 = S \checkmark$$

Estimating Sums

Suppose $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

and that

(i) $b_{n+1} \le b_n$ and $\lim_{n \to \infty} b_n = 0$

Then

$$|R_n| = |S - S_n| \le b_{n+1}$$

$$|R_{50}| = |S - S_{50}| \le b_{51}$$

Ex. Let $S = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n^3+1}$

Find the smallest N so

$$\underbrace{\left| S - \sum_{n=0}^{N} (-1)^n \cdot \frac{1}{n^3+1} \right|}_{|R_N|} < .001.$$

$$|R_N| \leq \frac{1}{(N+1)^3+1}$$

$$|R_1| \leq \frac{1}{2^3+1} = \frac{1}{9}$$

$$|R_2| \leq \frac{1}{3^3+1} = \frac{1}{28}$$

$$|R_8| \leq \frac{1}{9^3+1} = \frac{1}{730}$$

$$|R_9| \leq \frac{1}{10^3+1} = \frac{1}{1001} < .001.$$

$$\therefore \quad S = 1 - \frac{1}{2} + \frac{1}{9} - \frac{1}{28}$$

$$+ \frac{1}{65} - \frac{1}{126} + \frac{1}{217}$$

$$- \frac{1}{344} + \frac{1}{513} + \frac{1}{730} + R_9$$

If $S = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{5^n}$,

find the smallest N so

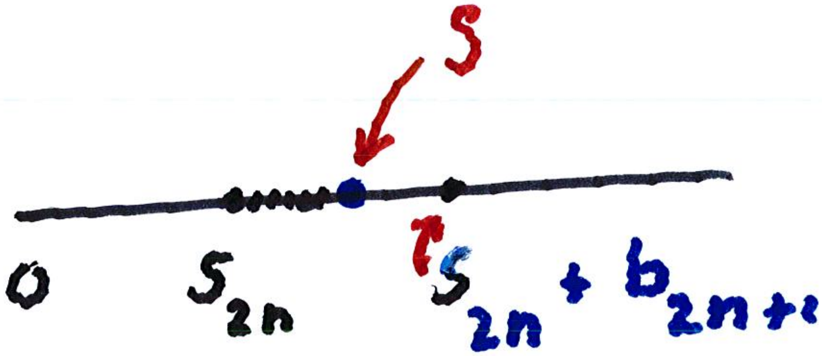
$$|R_N| = \left| S - \sum_{n=1}^{N} (-1)^n \cdot \frac{n^2}{5^n} \right| < 10^{-4}$$

$N = 8 \rightarrow |R_8| \leq \frac{81}{5^9} = .000041$

$N = 7 \rightarrow |R_7| \leq \frac{64}{5^8} = .00016$

$\therefore N = 8$

Why is the error estimate true?



$$\therefore |S - S_{2n}| \leq b_{2n+1}$$

OR $|S - S_N| \leq b_{N+1}$

$$|R_N| \leq |\text{first term omitted}|$$

i.e., where $n = N+1$.

It is known that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots$$

Calcuate e^{-1} to within .0001.

$$\frac{1}{6!} = \frac{1}{720} \qquad \frac{1}{7!} = \frac{1}{5040}$$

$$\frac{1}{8!} = \frac{1}{40320} < \frac{1}{10^4}$$

$$\therefore |R_7| \leq b_8 = \frac{1}{8!} < \frac{1}{10^4}$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} \cdots - \frac{1}{7!} + R_7$$

Approximate the sum of the series to within four decimal places. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n \cdot n!}$

$|R_N| < \frac{1}{2^{N+1}(N+1)!}$ want $< .0001 = \frac{1}{10,000}$

$N = 4: \quad \frac{1}{32 \cdot 120} \approx \frac{1}{3800}$

$N = 5 \quad \frac{1}{720 \cdot 64} \approx \frac{1}{46,000}$ ✓

$S_5 = \sum_{n=1}^{5} \frac{(-1)^n}{2^n \cdot n!}$