11.6 con't. Ratio and Root Test.

Ex. Find if $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

$$= \frac{2^{n+1} n!}{2^n (n+1)!} = \frac{2}{n+1} \to 0$$

as $n \to \infty$

$\therefore$ Series converges absolutely

Root Test

(Works well if a_n is an n-th power.)

(i) If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$,

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \underset{>1}{L}$ or $= \infty$,

then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L = 1$

then the test is inconclusive,

i.e., $\sum_{n=1}^{\infty} a_n$ may or may not converge.

Ex. Test the convergence of

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+3}\right)^n$$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\left(\frac{n-1}{2n+3}\right)^n\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left| \frac{n-1}{2n+3} \right|$$

$$= \lim_{n\to\infty} \frac{n\left(1-\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = \frac{1}{2} < 1$$

$\therefore$ Series converges. absolutely.

Ex. Consider $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\ln n+1}\right)^n$.

$$|a_n|^{\frac{1}{n}} = \left(\left(\frac{1}{\ln n+1}\right)^n\right)^{\frac{1}{n}}$$

$$= \frac{1}{\ln n+1} \to \frac{1}{\infty} = 0$$

$\therefore$ Series Converges

Ex. Study $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^{n^2}}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{2^{(n+1)^2}}{(n+1)!}}{\dfrac{2^{n^2}}{n!}}$$

$$= \frac{2^{2n+1}\, n!}{(n+1)!} = \frac{2^{2n+1}}{n+1} \to \infty$$

$$\lim_{x\to\infty} \frac{2^{2x+1}}{x+1} \overset{L}{=} \lim_{x\to\infty} \frac{(\ln 2)\, 2^{2x+1}\cdot 2}{1}$$

$$= \infty$$

$\therefore$ Series Diverges.

Rearrangements

A rearrangement of

$a_1 + a_2 + a_3 + \ldots$ would be

$a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \ldots$

Riemann showed that if

$\sum_{n=1}^{\infty} a_n$ converges conditionally,

and if r is any number, then

there is a rearrangent that

converges to r.

11.7 Strategy for Testing Series.

If $|r| < 1$, then the series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Compute $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \cdot 2^{n-1}}{3^{n-2}}$

$$a = \frac{2^{2-1}}{3^{2-2}} = 2 \qquad \therefore S = \frac{2}{1-\left(\frac{-2}{3}\right)}$$

$$r = \frac{-2}{3} \qquad = \frac{2}{5/3} = \frac{6}{5}$$

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2n-1}$ converge?

$\{2n-1\}$ is increasing

$\therefore$ $\frac{1}{2n-1}$ is decreasing, and converges to 0.

$\therefore$ Series converges conditionally

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n-1}{n^3+n}$

Let $f(x) = \frac{2x-1}{x^3+x}$

$$f'(x) = \frac{(x^3+x)\cdot 2 - (3x^2+1)(2x-1)}{(x^3+x)^2}$$

$$= \frac{-4x^3+3x^2+1}{(x^3+x)^2}$$

Diff. Num. $-12x^2+6x < 0$

if $x \geq 1$.

$\therefore -4x^3+3x^2+1 < 0$ if $x > 1$

$\therefore f'(x) < 0$ if $x > 1$.

$\rightarrow b_{n+1} < b_n$ if $n \geq 1$ $\therefore$ Series Converges Abs.

$$\text{Let } S = \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3n^2}.$$

Error Est

If $f(x) = \dfrac{2x+1}{3x^2}$,

$$\to f'(x) = \frac{3x^2 \cdot 2 - (2x+1)6x}{9x^4}$$

$$= \frac{-6x^2 - 6x}{9x^4} < 0 \quad \text{if } x \geq 1$$

$\therefore \dfrac{2n+1}{3n^2}$ is decreasing.

Compute Max. size of R_5

{using Alt. Series Error Estimate}

$$|S - S_5| = |R_5| < b_6$$

$$= \frac{13}{108}$$

Find smallest N so that

$$|S - S_N| < .1$$

$$|S - S_N| < b_{N+1}$$

$N = 7$ $\qquad b_8 = \dfrac{2 \cdot 8 + 1}{3 \cdot 8^2}$

$$= \frac{17}{192} < .1$$

But $N = 6$ $\qquad b_7 = \dfrac{2 \cdot 7 + 1}{149} > .1$

$\therefore N = 7$ $\qquad \dfrac{15}{149}$

Remember, for $\sum_{n=1}^{\infty} a_n$ to

converge, it must be that

$\lim_{n\to\infty} a_n = 0.$

For an algebraic function,

use the Limit Comparison

Test, where $\sum b_n$ is

a p-series.

Ex. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+2}}{2n^2+n}$

Set $b_n = \dfrac{\sqrt{n^3}}{n^2} = \dfrac{n^{3/2}}{n^2}$

$= \dfrac{1}{n^{1/2}}$

Assump $\lim \dfrac{a_n}{b_n} = L > 0$

$\therefore \displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$,

which diverges. $p = \frac{1}{2}$

$\therefore$ Original Series diverges

Ex. Use the Comparison Test

if $\sum a_n$ can be estimated

Ex. $\displaystyle\sum_{n=1}^{\infty} (1+\sin n)\cdot\frac{1}{1+2^n}$.

$$0 \le \frac{(1+\sin n)}{1+2^n} \le \cancel{\sum_{n=1}^{\infty}} \frac{2}{1+2^n}$$

$$\le 2\sum_{n=1}^{\infty}\frac{1}{2^n}$$

which is a

convergent geometric series.

Ex. Consider $\sum_{n=1}^{\infty} \frac{1}{n+2^n}$.

Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$0 \le \frac{1}{n+2^n} \le \frac{1}{2^n}.$$

Ex. What about $\sum_{n=2}^{\infty} \frac{1}{2^n - n}$

It is NOT TRUE that

$$0 \le \frac{1}{2^n - n} \underset{\substack{\uparrow \\ x}}{\le} \frac{1}{2^n}.$$

For this series, we should use the Limit Comparison Thm.

Set $a_n = \dfrac{1}{2^n - n}$

and $b_n = \dfrac{1}{2^n}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n - n}}{\frac{1}{2^n}}$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - n} = \lim_{n\to\infty} \frac{1}{1 - \frac{n}{2^n}} = 1$$

Ex. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

This series converges absolutely:

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges by Comp Test

(conv. by p-test, $p=2$)

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ conv. by Abs. Conv. Thm.

Ex. Consider $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$.

$$= \frac{3}{2}\sum_{k=1}^{\infty} \frac{2^k \cdot 3^k}{k^k}$$

$$= \frac{3}{2}\sum_{k=1}^{\infty} \left(\frac{6}{k}\right)^k$$

This converges by Root Test

$$|a_k|^{\frac{1}{k}} = \left[\left(\frac{6}{k}\right)^k\right]^{\frac{1}{k}}$$

$$= \frac{6}{k} \longrightarrow 0$$

as $k \to \infty$

Ex. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^2}$

Look at $\displaystyle\int_1^{\infty} x e^{-x}\,dx$

$$= \lim_{t\to\infty} \int_1^{t} x e^{-x}\,dx$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{t\to\infty} \left(-xe^{-x} \Big|_1^t + \int_1^t e^{-x}\,dx \right)$$

$$\lim_{t\to\infty} \left(-t e^{-t} + 1\cdot e^{-1}\right) - e^{-x}\Big|_1^t$$

$$\to e^{-1} + e^{-1} = 2e^{-1}$$