11.8 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a variable and the c_n's are the coefficients of the series.

A power series may converge for some values of x and diverge for others.

If we set $c_n = 1$ for all n, we obtain

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots,$$

which converges to $\frac{1}{1-x}$ if $|x| < 1$

and diverges if $|x| \geq 1$.

More generally the series

$$(1) \quad \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1 (x-a) + c_2 (x-a)^2$$

$$+ \dots + c_n (x-a)^n + \dots$$

is called a series <u>in $(x-a)$</u>

or a power series <u>centered</u>

<u>at a</u> or <u>about a</u>. The

series in (1) obviously

converges when $x = a$.

we often use the

Ratio Test to determine

for which x the series

converges.

Ex. For which x does

$$\sum_{n=0}^{\infty} \underbrace{n!\, x^n}_{a_n} \quad \text{converge?}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= (n+1)\,|x| \to \infty \quad \text{if } x \neq 0.$$

Therefore $\sum_{n=0}^{\infty} n!\, x^n$ only

converges if $x = 0$.

Ex. For which x does

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+2)\,2^n} \quad \text{converge ?}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{x^{n+1}}{(n+3)\,2^{n+1}}}{\dfrac{x^n}{(n+2)\,2^n}} \right|$$

$$= |x| \frac{(n+2)\,2^n}{(n+3)\,2^{n+1}} = \frac{|x|}{2} \frac{(n+2)}{(n+3)}$$

$$\to \frac{|x|}{2} \quad \text{as } n \to \infty \qquad (\text{if } x \neq 0).$$

$$\therefore \sum_{n=1}^{\infty} \frac{x^n}{(n+2)\,2^n} \quad \text{converges}$$

if $\dfrac{|x|}{2} < 1$, i.e. if $|x| < 2.$

Also the series diverges

if $\dfrac{|x|}{2} > 1$, i.e., if $|x| > 2$

If $x = 2$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2}$

which diverges

If $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+2)\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)},$$

which converges by the Alternating Series Test.

Thus the series converges if $-2 \leq x < 2$ and diverges elsewhere.

Ex. For which x does

$$\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+1)^2} \text{ converge?}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{2^{n+1}(x-2)^{n+1}}{(n+2)^2}}{\dfrac{2^n (x-2)^n}{(n+1)^2}} \right|$$

$$= \left| \frac{2^{n+1}(x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{2^n (x-2)^n} \right|$$

$$= 2|x-2| \cdot \frac{(n+1)^2}{(n+2)^2} \to 2|x-2|$$

as $n \to \infty$

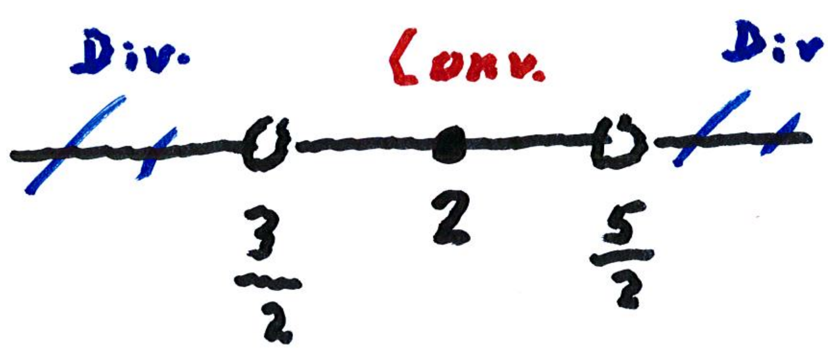
$\therefore$ If $2|x-2| < 1$, i.e. if $|x-2| < \frac{1}{2}$

then the series converges.

Also, if $2|x-2| > 1$, i.e., if

$|x-2| > \frac{1}{2}$. then the series

diverges



If $x = \frac{5}{2}$, the series is

$$\sum_{n=1}^{\infty} \frac{2^n \cdot \left(\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges

If $x = \frac{3}{2}$, then $(x-2)^n = \left(-\frac{1}{2}\right)^n$,

so the series is

$$\sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}$$

which also converges.

Thus the series when

$$\frac{3}{2} \le x \le \frac{5}{2}.$$

Ex. Find all x where

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad (0! = 1)$$

converges.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$$

$$= \left| \frac{x^{n+1}}{(n+1)!} \right| \cdot \left| \frac{n!}{x^n} \right|$$

$$= \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

Given any power series, it always converges in a symmetric open interval.

Thm. For a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$ there are

only 3 possibilities:

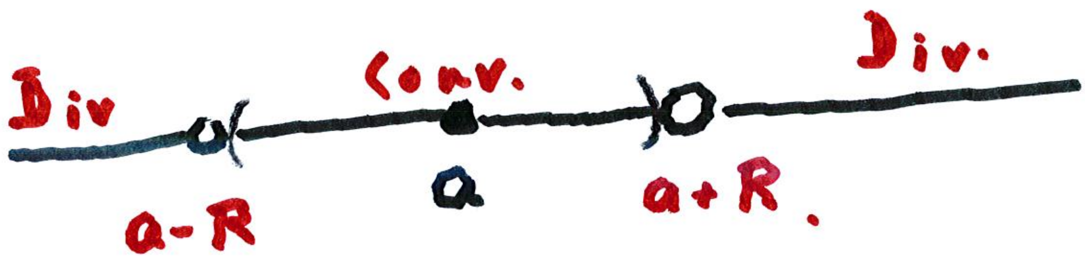
(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a number $R > 0$ so that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$

The number R in (iii) is called the **Radius of Convergence.**

In case (i), we set $R = 0$, and in case (ii), we set $R = \infty$



The above theorem says nothing about the cases when

$x = a - R$ or $x = a + R$.

So far, we have

Series	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	1	$(-1, 1)$
$\sum_{n=0}^{\infty} n!\, x^n$	0	$\{0\}$
$\sum_{n=1}^{\infty} \frac{x^n}{(n+2)2^n}$	2	$[-2, 2)$
$\sum_{n=1} \frac{2^n (x-2)^n}{(n+1)^2}$	$\frac{1}{2}$	$\left[\frac{3}{2}, \frac{5}{2}\right]$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞	$(-\infty, \infty)$

The advantage of power series is that we can greatly enlarge the set of functions.

Ex. Find the set of convergence of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+2}}{2^{2n+2} (n+1)!^2} \cdot \frac{2^{2n} (n!)^2}{x^{2n}} \right|$$

$x \neq 0$

$$= \frac{|x|^2}{2^2} \cdot \frac{1}{(n+1)^2} \to 0 \text{ as } n \to \infty.$$

Hence the power series converges for all x, and so $R = \infty$.

Find the set of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+4}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}\,|x|^{n+1}}{\sqrt{n+5}} \cdot \frac{\sqrt{n+4}}{2^n\,|x|^n}$$

$$= 2|x|\sqrt{\frac{n+4}{n+5}} \to 2|x|.$$

$\therefore$ Series converges if $2|x| < 1$

$\to |x| < \frac{1}{2}$.

and Series diverges if $2|x| > 1$

$\to |x| > \frac{1}{2}$

Look at endpoints

$x = \frac{1}{2}$ and $x = -\frac{1}{2}$.

$$x = \frac{1}{2} \rightarrow \sum_{n=0}^{\infty} \frac{(-2)^n \frac{1}{2^n}}{\sqrt{n+4}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

which converges by the

Alt. Series Test

When $x = -\frac{1}{2}$,

$$\rightarrow \sum_{n=0}^{\infty} \frac{(-2)^n \cdot \left(-\frac{1}{2}\right)^n}{\sqrt{n+4}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$, which diverges

by the Limit Comp. Test.

$R = \frac{1}{2}$, Int. of Conv.

is $\left(-\frac{1}{2}, \frac{1}{2}\right]$

Why is the Interval of Convergence symmetric?

Suppose $\sum_{n=0}^{\infty} c_n r^n$ converges.

Then the sequence $|c_n r^n|$ is bounded.

$\therefore$ There is a constant M so $|c_n r^n| \le M$.

Now let x be any number with $|x| < r$. Then

$$\sum_{n=0}^{\infty} |c_n x^n| \le \sum_{n=0}^{\infty} |c_n r^n| \cdot \left|\frac{x}{r}\right|^n$$

$$\le \sum_{n=0}^{\infty} M \cdot \left|\frac{x}{r}\right|^n$$

$$= M \cdot \frac{1}{1 - \left|\frac{x}{r}\right|} .$$

(since the series is a geometric series.)

Thus the series $\sum_{n=0}^{\infty} c_n x^n$

converges (by the Abs. Conv. Test)

If we can find r to be arbitrarily large, then

$\sum_{n=0}^{\infty} c_n x^n$ conv. for all x.

Otherwise, we can find r

as close as we want to R