11.10 Taylor Series cont'd.

Find the Maclaurin Series of

$$\frac{\tan^{-1}x}{x}.$$

Recall we have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \qquad |x|<1$$

Divide by x:

$$\frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots$$

if $|x|<1$.

Ex. Find the Taylor series of

$f(x) = \cos x$ about $a = \frac{\pi}{4}$

$f(x) = \cos x$ $\quad f(\pi/4) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x$ $\quad f'(\pi/4) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos x$ $\quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin x$ $\quad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \cos x$

(same as $f(x)$)

$$\cos x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\frac{\sqrt{2}}{2}}{2!}\left(x - \frac{\pi}{4}\right)^2$$

$$+ \frac{\frac{\sqrt{2}}{2}}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{\frac{\sqrt{2}}{2}}{4!}\left(x - \frac{\pi}{4}\right)^4 - \dots$$

or

$$\cos x = \frac{\sqrt{2}}{2}\left(1 - \left(x - \frac{\pi}{4}\right) - \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}\right.$$

$$\left. + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!} - \dots\right)$$

Ex. Find the Taylor series of $\ln x$ about $a = 2$.

Ex.

$$f(x) = \ln x \qquad f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(2) = -\frac{1}{2^2}$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \cancel{-}\frac{2}{2^3}$$

$$f^{(4)}(x) = \frac{-3\cdot 2}{x^4} \qquad f^{(4)}(2) = \frac{-3\cdot 2}{2^4}$$

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

$$f^{(n)}(2) = \frac{(-1)^{n-1} (n-1)!}{2^n}$$

$$\therefore \ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{\frac{1}{2^2}(x-2)^2}{2!}$$

$$- \frac{2}{2^3 \cdot 3!} (x-2)^3$$

$$+ \frac{3 \cdot 2}{2^4 \cdot 4!} (x-2)^4$$

$$\cdots + \frac{(-1)^{n-1} (n-1)!}{2^n \cdot n!} (x-2)^n + \cdots$$

Therefore

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{2^n \cdot n}$$

Find the ~~Mac.~~ Taylor Series of $f(x) = \frac{1}{\sqrt{x}}$ about $a = 4$

$f(x) = \frac{1}{\sqrt{x}}$ $\qquad$ $f(4) = \frac{1}{2}$

$f'(x) = \left(-\frac{1}{2}\right) x^{-\frac{3}{2}}$ $\qquad$ $f'(4) = -\frac{1}{2} \cdot \frac{1}{2^3}$

$f''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-\frac{5}{2}}$ $\qquad$ $f''(4) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \frac{1}{2^5}$

$$f'''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-\frac{7}{2}}$$

$$f'''(4) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \frac{1}{2^7}$$

$$\vdots$$

$$f^{(n)}(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)\cdot x^{\frac{-2n-1}{2}}$$

$$\therefore f^{(n)}(4) = \left(-\frac{1}{2}\right)\cdots\left(\frac{1-2n}{2}\right)\cdot 4^{-\frac{(2n+1)}{2}}$$

or

$$f^{(n)}(4) = \left(\frac{-1}{2}\right)\cdots\left(\frac{1-2n}{2}\right)\cdot \frac{1}{2^{2n+1}}$$

Hence,

$$\frac{1}{\sqrt{x}} = \frac{1}{2} + \sum_{n=1}^{\infty} \underbrace{\frac{\left(\frac{-1}{2}\right)\cdots\left(\frac{1-2n}{2}\right)\cdot\frac{1}{2^{2n+1}}}{n!}(x-4)^n}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{-1-2n}{2}\right)\cdot\frac{1}{2^2}\cdot(x-4)}{n+1}\right|$$

$$\rightarrow \frac{|x-4|}{4}$$

$\therefore$ Radius of Convergence equals 4.

Ex.

Let $f(x) = (1+x)^k$, where k is a given real number. Find the Maclaurin series of f.

$f(x) = (1+x)^k$ $\qquad$ $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad$ $f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$

$f'''(0) = k(k-1)(k-2)$

$$f^{(n)}(x) = k(k-1)\dots(k-n+1)(1+x)^{k-n+1}$$

$$f^{(n)}(0) = k(k-1)\dots(k-n+1)$$

$$\therefore (1+x)^k = 1 + \frac{kx}{1!} + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3$$

$$+ \dots + \frac{k(k-1)\dots(k-n+1)}{n!}x^n + \dots$$

If we set $\binom{k}{n} = \dfrac{k(k-1)\dots(k-n+1)}{n(n-1)\dots 2\cdot 1}$,

then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\binom{k}{0} = 1$.

This is called the Binomial Series

When k is a positive number,

one has

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n .$$

so that when $a=1$ and $b=x$,

we get

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n ,$$

$$\left(\text{since } \binom{k}{n} = 0 \text{ if } n \geq k+1 \right)$$

Ex. Use the Binomial Series with $k = -\frac{1}{2}$ to find the Maclaurin Series of $\frac{1}{\sqrt{4-3x}}$

$$\frac{1}{\sqrt{4-3x}} = \frac{1}{\sqrt{4\left(1-\frac{3x}{4}\right)}}$$

$$= \frac{1}{2}\left(1-\frac{3x}{4}\right)^{-\frac{1}{2}}.$$

Using the series with $k = -\frac{1}{2}$ we have:

$$(1+u)^{-\frac{1}{2}}$$

$$= 1 + \frac{\left(-\frac{1}{2}\right)u}{1!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)u^2}{2!}$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)u^3}{3!} + \cdots \quad (1)$$

If we set $u = -\frac{3x}{4}$, then

$$\frac{1}{2}\frac{1}{\sqrt{1-\frac{3x}{4}}} = \frac{1}{2}\left\{1 + \frac{\frac{1}{2}\left(\frac{3x}{4}\right)}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\left(\frac{3x}{4}\right)^2}{2!}\right.$$

$$\left. + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3x}{4}\right)^3}{3!} + \cdots\right\}$$

f we set $u = -x^2$ in (1),

then we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!} x^4$$

$\frac{1}{\sqrt{1+u}}$

$u = -x^2$

$$+ \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) x^6}{3!} + \cdots$$

$$= 1 + \frac{x^2}{2} + \frac{1\cdot 3}{2\cdot 4} x^4$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} x^6 + \cdots$$

Recall that

$$\frac{d}{dx}\{\sin^{-1}x\} = \frac{1}{\sqrt{1-x^2}}$$

which gives

$$\sin^{-1}x = C + \int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1}x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5}$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^7}{7} + \cdots$$

Ex. Find the Maclaurin series

of $\frac{\sin x}{x}$, and

find $\int_0^{1/2} \frac{\sin t}{t}\, dt$

$$\frac{\sin t}{t} = \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots}{t}$$

$$= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

$$\int_0^{\frac{1}{2}} \frac{\sin t}{t}\, dt = \int_0^{\frac{1}{2}} 1 - \frac{t^3}{3!} + \frac{t^4}{5!} - \dots dt$$

$$= t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \dots \Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 3 \cdot 6} + \frac{1}{32 \cdot 5 \cdot 120}$$

$$- \underbrace{\frac{1}{128 \cdot 7 \cdot 7!}}_{\le \frac{1}{128 \cdot 7 \cdot 5040} < 10^{-6}}$$

$$\therefore \left| S - \left(\frac{1}{2} - \frac{1}{144} + \frac{1}{19200} \right) \right| < 10^{-8}$$

Ex. Find the Maclaurin Series of $\dfrac{e^x - 1 - x}{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$