# 12.3 Dot Product

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, in $\mathbb{R}^3$

then $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Ex $\langle 3, -1 \rangle \cdot \langle 2, 4 \rangle$ in $\mathbb{R}^2$

$$= 3\cdot 2 + (-1)\cdot 4 = \underline{\underline{2}}$$

## Properties of the Dot Product

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

3 $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

5. $\vec{0} \cdot \vec{a} = 0$

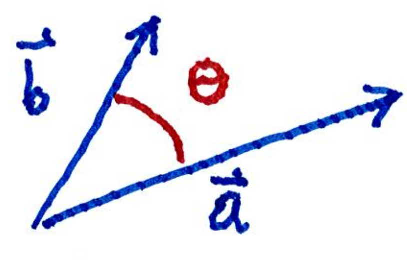
Note the dot product is a number.

## Geometric Meaning of the dot product:

Given 2 vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

or $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$




$\theta$ = angle between $\vec{a}$ and $\vec{b}$

$$0 \leq \theta \leq \pi$$

Ex. Find angle $\theta$ between

$\vec{a} = \langle 2, 1, 1 \rangle$ and $\vec{b} = \langle 1, 3, -1 \rangle$

$$\vec{a} \cdot \vec{b} = 2 + 3 - 1 = 4$$

$$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\therefore \cos\theta = \frac{4}{\sqrt{6}\sqrt{11}} = \frac{4}{\sqrt{66}}$$

Apply $\cos^{-1}$ to both sides

$$\cos^{-1}(\cos\theta) = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right)$$

or $$\theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right)$$

Ex. Compute the angle between

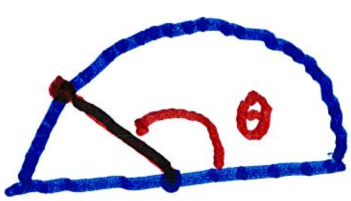
$\vec{a} = \langle 1, 2 \rangle$ and $\vec{b} = \langle 1, -3 \rangle$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1-6}{\sqrt{1+4}\,\sqrt{1+9}}$$

$$= \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-5}{\sqrt{50}}$$

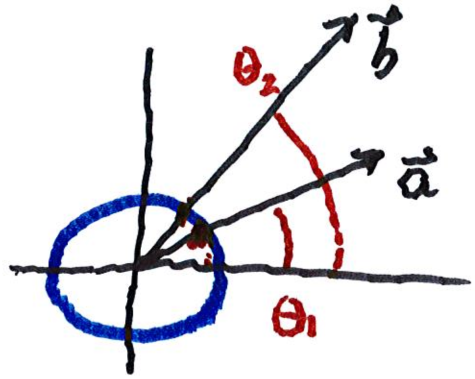
$$= \frac{-5}{5\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\therefore \cos\theta = -\frac{1}{\sqrt{2}}$$




$$\theta = \frac{3\pi}{4}$$

Here's why:




If $\vec{a}_1 = R_1 \langle \cos\theta_1 , \sin\theta_1 \rangle$ .

then $|\vec{a}| = R_1$

If $\vec{b} = R_2 \langle \cos\theta_2 , \sin\theta_2 \rangle$

then $|\vec{b}| = R_2$

$$\vec{a} \cdot \vec{b} = R_1 R_2 \{ \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \}$$

$$= R_1 R_2 \cos(\theta_1 - \theta_2)$$

$$= |\vec{a}||\vec{b}| \cos\theta$$

∴ 2 nonzero vectors

$\vec{a}$ and $\vec{b}$ are perpendicular

if and only $\underline{\vec{a} \cdot \vec{b} = 0}$

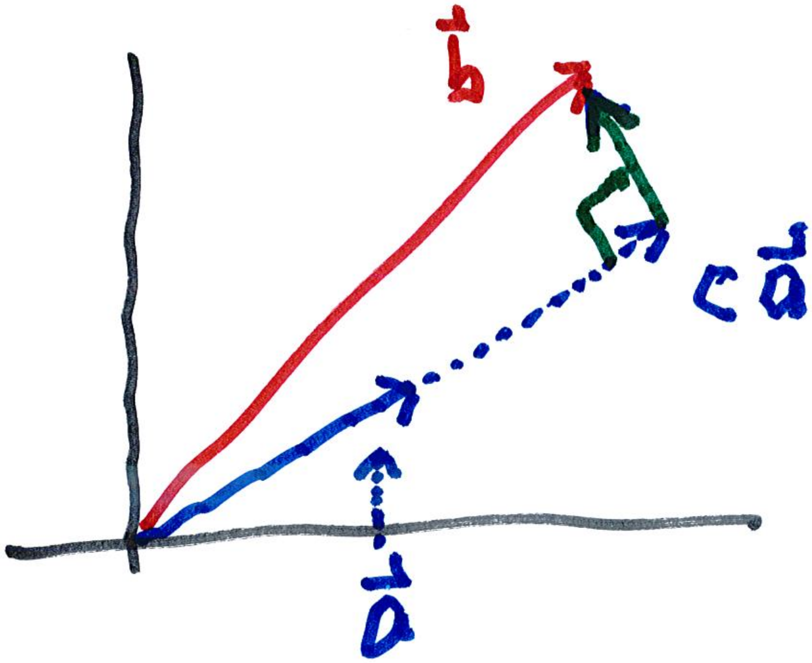
Ex. Show $\vec{a} = 2\vec{i} + \vec{j} + 4\vec{k}$

and $\vec{b} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ are

perpendicular (orthogonal)

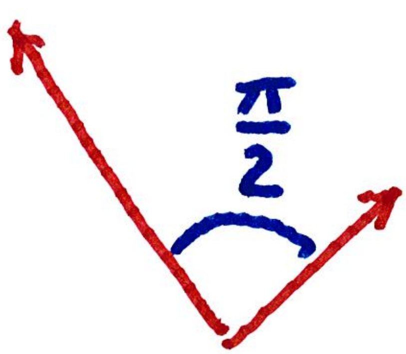
$\vec{a} \cdot \vec{b} = 6 + 2 - 8 = 0$ ✓

Projections. Given 2 vectors $\vec{a}$ and $\vec{b}$, find a number $c$ so that $\vec{b} - c\vec{a}$ is $\perp$ to $\vec{a}$




We need $(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$

We say 2 nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) if the angle between them is $\frac{\pi}{2}$. that is $\cos\theta = 0$. But this happens if $\vec{a} \cdot \vec{b} = 0$.



i.e., $\vec{b}\cdot\vec{a} - c\,\vec{a}\cdot\vec{a} = 0$

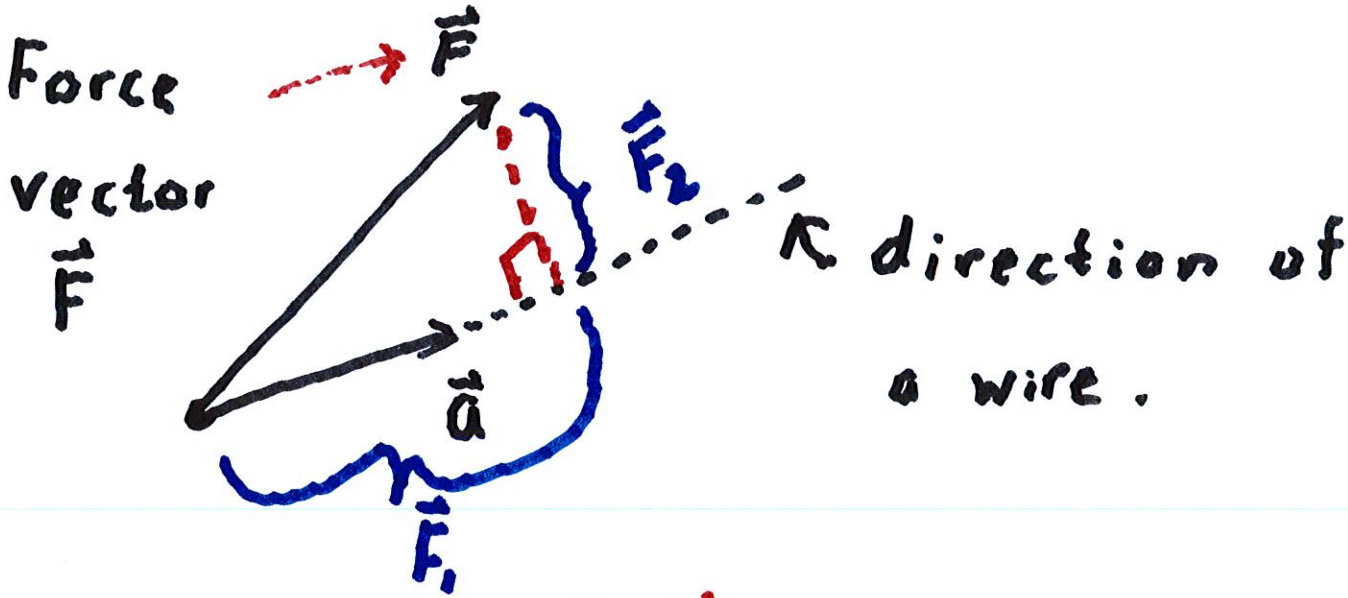
or $\vec{b}\cdot\vec{a} = c|\vec{a}|^2$

or $\boxed{c = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}}$

Using this c we have written $\vec{b}$ as a sum:

$$\vec{b} = c\vec{a} + (\vec{b} - c\vec{a}),$$

where these 2 are $\perp$




How much of $\vec{F}$ points in

the direction of $\vec{a}$ ?

Thus $\vec{F} = \vec{F}_1 + \vec{F}_2$ ,

where $\vec{F}_1$ is $\perp$ to $F_2$

and $\vec{F}_1$ is a multiple of $\vec{a}$ .

$\vec{F}_1$ points along the wire $\vec{a}$.

$$\text{proj}_{\vec{a}} \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

C

is the vector projection

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$= \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}\,\frac{\vec{a}}{|\vec{a}|}$$

This is the component of $\vec{b}$ in the $\vec{a}$ direction

$$\boxed{\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

(the scalar projection)

Ex Find $\text{proj}_{\vec{a}} \vec{b}$ if

$\vec{a} = \langle 2, 1, 1 \rangle$ and $\vec{b} = \langle 1, 3, -2 \rangle$

$$\left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

$$\vec{a} \cdot \vec{b} = 2 + 3 - 2$$
$$= 3$$

$$|\vec{a}|^2 = 4 + 1 + 1 = 6$$

$$\therefore \text{proj}_{\vec{a}} \vec{b} = \frac{3}{6} \langle 2, 1, 1 \rangle$$

$$= \langle 1, \tfrac{1}{2}, \tfrac{1}{2} \rangle$$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3}{\sqrt{6}}$$

This is the size of

$\text{proj}_{\vec{a}}\vec{b}$ ( It's negative if

the angle between $\vec{a}$ and $\vec{b}$

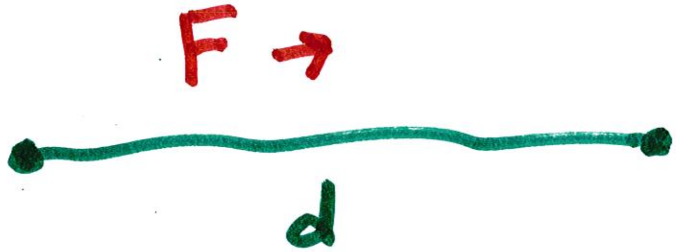
is $> \frac{\pi}{2}$.

In physics if a force acts on an object moving on a line, then the work is

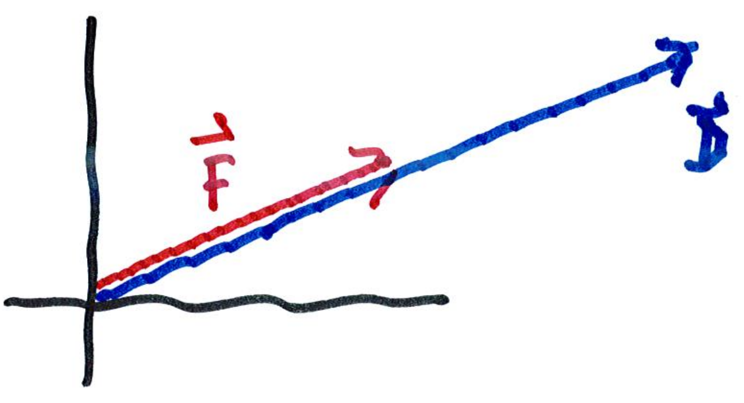
$$W = F\,d$$

magnitude of force

distance object moves



Suppose a force of magnitude $F$ moves an object $d$ units (in the same dir. as force), then work $= W = Fd$

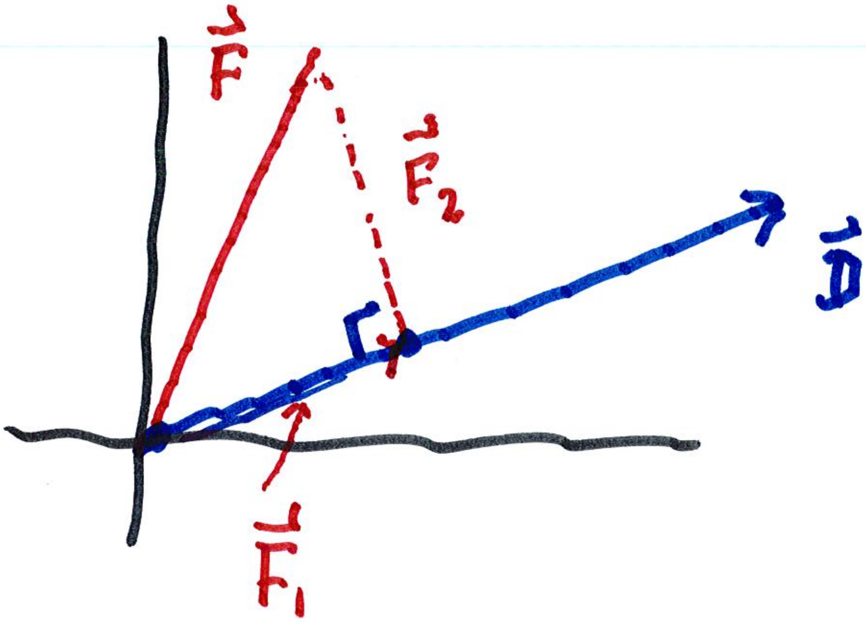



$d = |\vec{D}|$

$$W = |\vec{F}||\vec{D}|$$

or $-|\vec{F}||D|$

Suppose $\vec{F}$ acts in a different direction



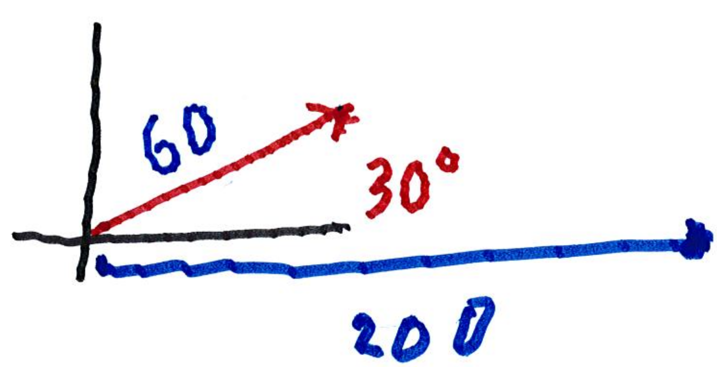

Magnitude of $\vec{F}_1$ is $\dfrac{\vec{F}\cdot\vec{D}}{|\vec{D}|}$

$$\therefore W = |F_1||\vec{D}| = \frac{\vec{F}\cdot\vec{D}}{|\vec{D}|}|\vec{D}|$$

$$= \vec{F}\cdot\vec{D}$$

$$\therefore \boxed{W = \vec{F} \cdot \vec{D}}$$

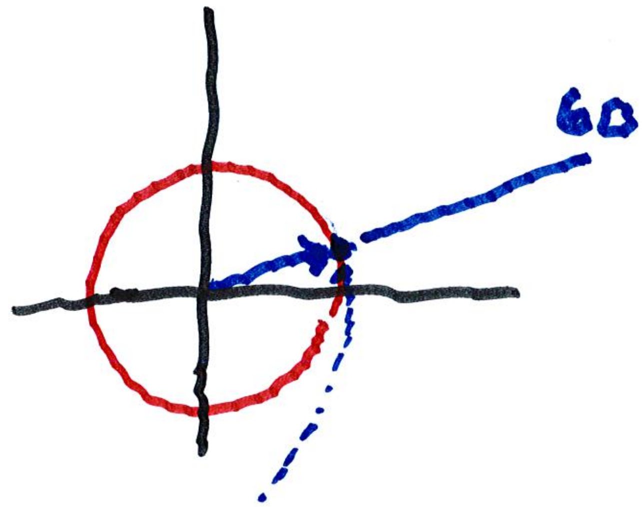
Ex. A wagon is pulled a distance of 200 m along a level path by a force of 60 N. If the handle is held at an angle of 30° above the horizontal, find the work done.



$$\vec{D} = \underline{200\,\vec{i} + 0\,\vec{j}}$$

$$\vec{F} = 60\left(\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}\right)$$

$$\vec{F} = \underline{30\sqrt{3}\,\vec{i} + 30\,\vec{j}}$$




$$\cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\theta = \frac{\pi}{6}$$

$$\rightarrow \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$$

$$W = \vec{F} \cdot \vec{D}$$

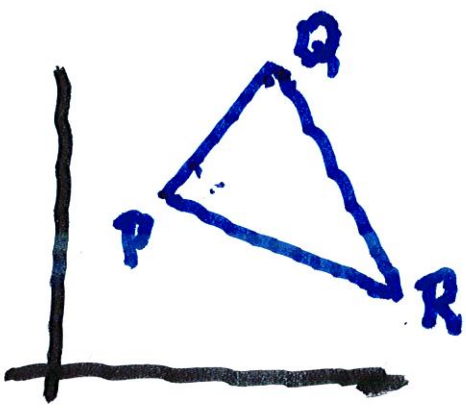
$$= 30\sqrt{3} \cdot 200 + 0$$

$$= \underline{\underline{6000\sqrt{3}\ J}}$$

Ex. Is the triangle with vertices at

$P(1,-3,-2)$ $Q(2,0,-4)$ and

$R(6,-2,-5)$ a right triangle?




Look at $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

If not, look at $\overrightarrow{QP}$ and $\overrightarrow{QR}$

etc.

$$\overrightarrow{QP} = \langle 1-2,\ -3-0,\ -2-(-4) \rangle$$

$$= \langle -1,\ -3,\ 2 \rangle$$

$$\overrightarrow{QR} = \langle 6-2,\ -2-0,\ -5-(-4) \rangle$$

$$= \langle 4,\ -2,\ -1 \rangle$$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = (-1)\cdot 4 + (-3)(-2)$$

$$+ 2(-1)$$

$$= -4 + 6 - 2 = 0$$

$\therefore$ These are orthogonal.