Ex. Find the Maclaurin series

of $\frac{\sin x}{x}$, and

find $\int_0^{1/2} \frac{\sin t}{t}\, dt$

$$\frac{\sin t}{t} = \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots}{t}$$

$$= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

$$\int_0^{\frac{1}{2}} \frac{\sin t}{t}\, dt = \int_0^{\frac{1}{2}} 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots\, dt$$

$$= t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \dots \Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 3 \cdot 6} + \frac{1}{32 \cdot 5 \cdot 120}$$

$$- \underbrace{\frac{1}{128 \cdot 7 \cdot 7!}}_{\leq \frac{1}{128} \cdot \frac{1}{7} \cdot \frac{1}{5040} < 10^{-6}}$$

$$\therefore \left| S - \left(\frac{1}{2} - \frac{1}{144} + \frac{1}{19200} \right) \right| < 10^{-8}$$

Ex. Find the Maclaurin Series of $\dfrac{e^x - 1 - x}{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$

Using the Binomial Series with

$k = -\frac{1}{2}$, we get

$$(1+u)^{-\frac{1}{2}} = 1 + \frac{\left(-\frac{1}{2}\right)}{1!} u + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} u^2$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} u^3 + \cdots$$

Setting $u = -x^2$, we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} x^8 + \cdots$$

$$\sin^{-1} x = \int \frac{dx}{\sqrt{1-x^2}}$$

$$= x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\,\frac{x^9}{9} + \cdots + C$$

Ex. Find an approximation

of $\sqrt[3]{8+x} = \left(8\left(1+\frac{x}{8}\right)\right)^{\frac{1}{3}}$

$$= 2\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$$

First, find $(1+u)^{\frac{1}{3}}$

$$= 1 + \frac{u}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{1\cdot 2}u^2$$

$$+ \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\cdot 2\cdot 3}u^3$$

$$= 1 - \frac{(-1)\cdot}{3}u + \frac{(-1)\cdot 2}{3\cdot 6}u^2$$

$$- \frac{(-1)\cdot 2\cdot 5}{3\cdot 6\cdot 9}u^3 + \frac{(-1)\cdot 2\cdot 5\cdot 8\cdot}{3\cdot 6\cdot 9\cdot 12}u^4$$

Now set $v = \frac{x}{8}$ and multiply

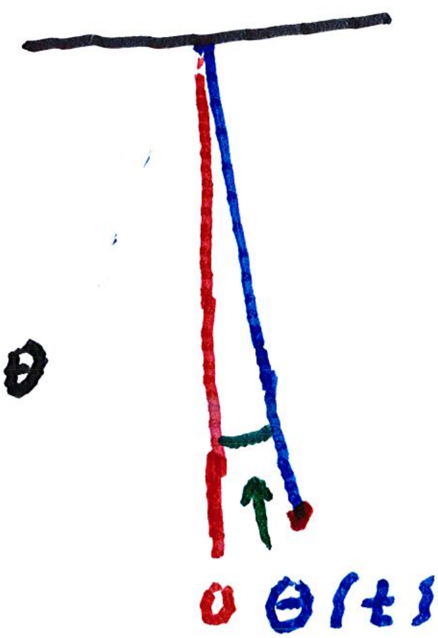
by 2:

$$2\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\left(1 - \frac{(-1)}{3}\cdot\frac{x}{8} + \frac{(-1)\cdot 2}{3\cdot 6}\left(\frac{x}{8}\right)^2\right.$$

$$- \frac{(-1)\cdot 2\cdot 5}{3\cdot 6\cdot 9}\left(\frac{x}{8}\right)^3 + \frac{(-1)\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\left(\frac{x}{8}\right)^4$$

$$\left. - \dots \right)$$

In physics and mechanics, it's often necessary to solve differential equations, such as



$$\frac{d^2\theta}{dt^2} = -k^2 \sin\theta$$

where $k^2 = \dfrac{g}{\ell}$

If the angle θ is small,

we can approximate $\sin\theta$

$$\sin\theta = \theta \underbrace{- \frac{\theta^3}{6} + \frac{\theta^5}{120}}_{\text{small}}$$

$$\therefore \frac{d^2\theta}{dt^2} = -k^2\theta.$$

This is much easier to solve.

Ex. Find the Maclaurin series of $\cosh x$.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

$$\therefore \cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots$$

$$\frac{\frac{1}{4!} + \frac{1}{4!}}{2} = \frac{1}{4!}$$

Similarly, to find the Maclaurin series of sinh x,

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)}{2}$$

$$= \frac{0 + 2x + 0 + \frac{2x^3}{3!} + 0}{2}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Find the sum of the series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$$

$$e^{-u} = 1 - \frac{u}{1!} + \frac{u^2}{2!} - \frac{u^3}{3!} + \cdots$$

Now set $u = x^4$

$$\rightarrow e^{-x^4} = 1 - \frac{x^4}{1!} + \frac{x^{4\cdot 2}}{2!} - \frac{x^{12}}{3!} - \cdots$$

$$\therefore \text{Series} = e^{-x^4}$$

Evaluate $\int_0^1 e^{-x^4}dx$ to

within .001.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

Set $u = -x^4$

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$$

$$\therefore \int_0^1 e^{-x^4}\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{4n}}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{4n+1} \cdot \frac{1}{n!}$$

$$1 - \frac{1}{5} + \frac{1}{9}\cdot\frac{1}{2} - \frac{1}{13}\cdot\frac{1}{6}$$

$$+ \frac{1}{17}\cdot\frac{1}{24} - \frac{1}{21}\cdot\frac{1}{120} + \dots$$

Series is alternating, terms are decreasing. $\frac{1}{21}\cdot\frac{1}{120} < .001$

$\therefore$ Use first 5 terms.

Why not use Taylor series to compute the approximate value for e^{10}.

1. Not all functions have a power series.

2. It might be many terms are needed to get a close approximate function.

Ex. $f(x) = e^x$.

To compute e^{10}, how big must n be?

$$e^{10} = 1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots + \frac{10^n}{10!} + \cdots$$

Let $n = 20$

$$\frac{10}{1} \cdot \frac{10}{2} \cdot \frac{10}{3} \cdots \frac{10}{n} \cdot \cdots \frac{10}{20}$$

Still > 1.