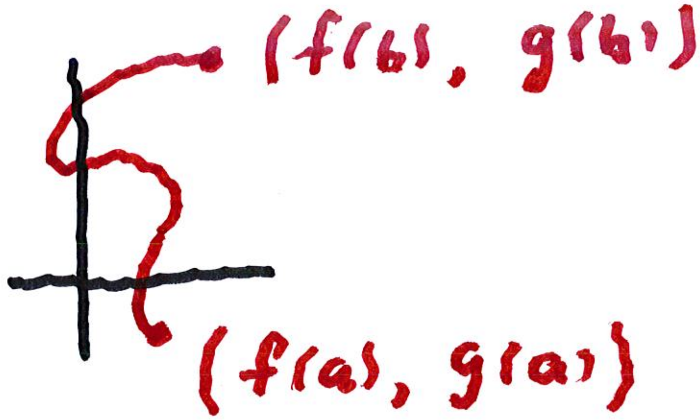
10.1 Curves Defined by Parametric Equation.

Suppose that a particle in the plane has location at

$(f(t), g(t))$, for $a \le t \le b$.



The set of all points $(f(t), g(t))$ defines a parametric curve C.

Ex. Suppose C is defined by

$$x = t^2 + 2t, \qquad y = t - 1,$$

$$\text{where } -3 \le t \le 2.$$

To visualize C, it's useful if we can solve for the parameter t. $\quad t = y + 1$

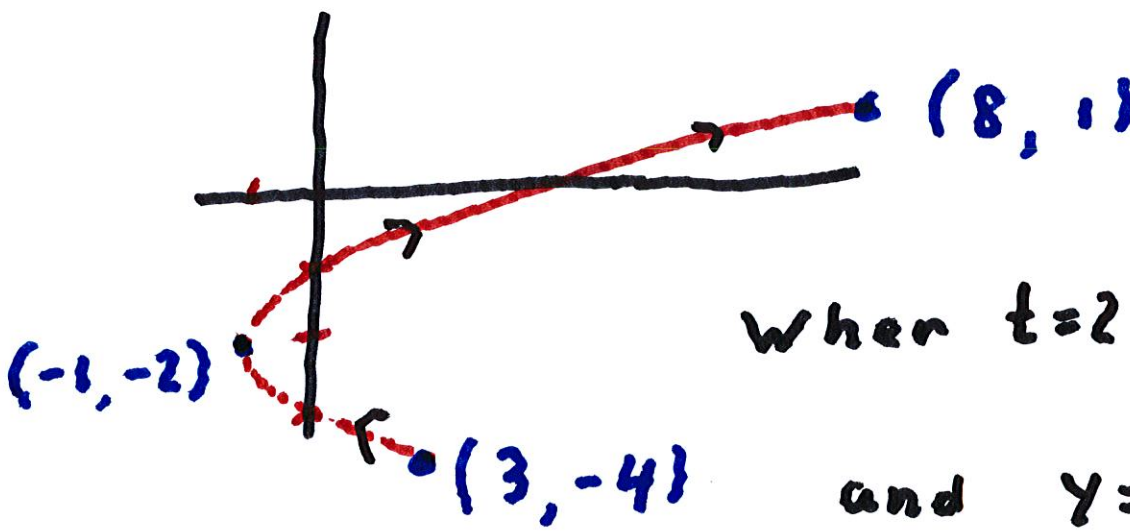
$$\to \quad x = (y+1)^2 + 2(y+1)$$

$$x = y^2 + 4y + 3.$$

$$x = (y+2)^2 - 1$$

This defines a parabolic arc

with a vertex at $(-1, -2)$



When $t=2$, $x=8$

and $y=1$

When $t=-3$,

$x=3$ and

$y=-4$

As t moves from -3 to 2,

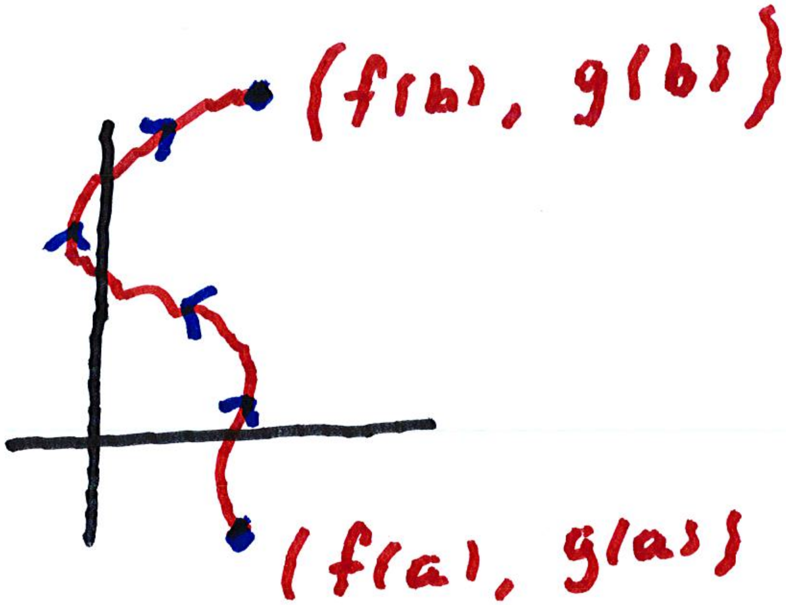
the particle moves from

$(3, -4)$ to $(8, 1)$.

In general, if $x = f(t)$ and $y = g(t)$,

the particle moves from

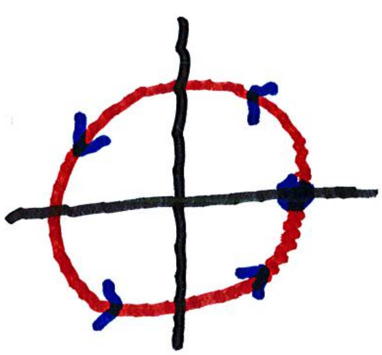
$(f(a), g(a))$ to $(f(b), g(b))$.



The circle of radius R can be parameterized by

$x = R\cos t, \quad y = R\sin t$

for $0 \le t \le 2\pi$



Consider the curve

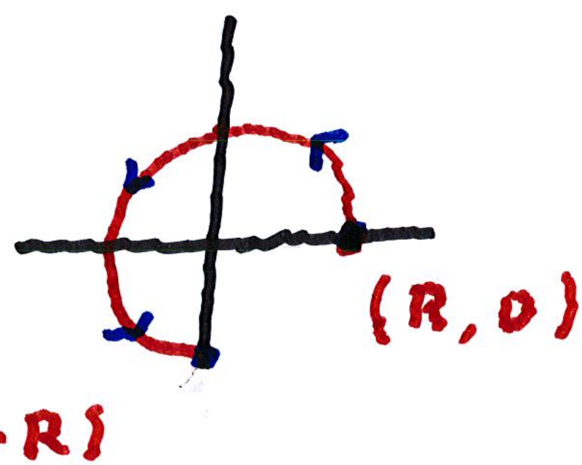
$$x = R\cos 3t, \quad y = R\sin 3t,$$

$$\text{for } 0 \le t \le \frac{\pi}{2}.$$

The initial point is $(R, 0)$

and the terminal point is

$(0, -R)$



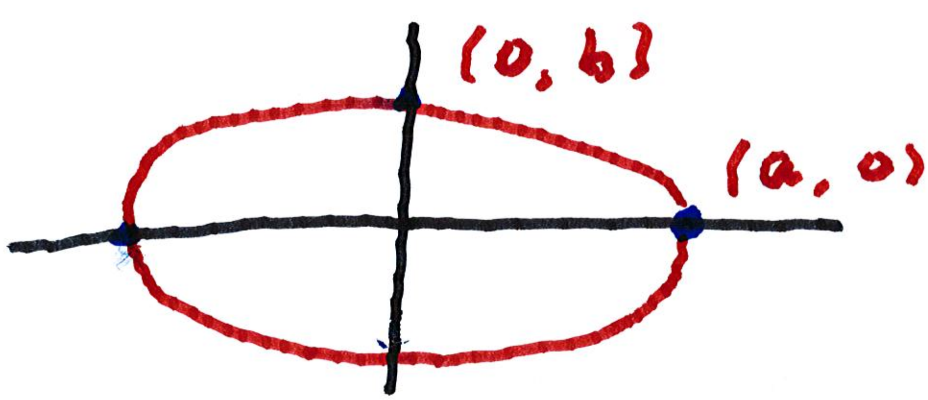
Ex. The equations

$$x = a\cos t\,, \quad y = b\sin t \quad 0 \le t \le 2\pi$$

define an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2}$$

$$= \cos^2 t + \sin^2 t = 1$$



Ex. Consider the curve

$$x = \cos t, \quad y = \cos^2 t$$

$$0 \le t < \infty$$

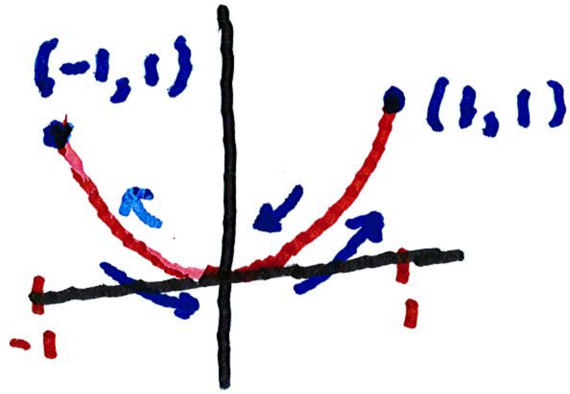
This satisfies $y = x^2$

from $(1,1)$ to $(-1,1)$

$t = 0$ $\qquad$ $t = \pi$

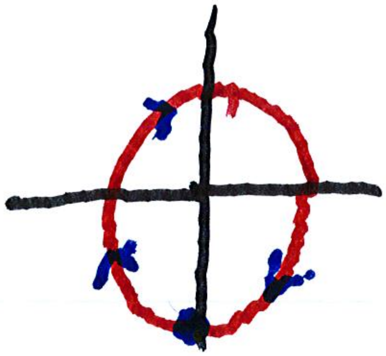
and then back to $(1,1)$.

etc.



As $t \to \infty$, the particle goes through one cycle as t goes from $2n\pi$ to $2(n+1)\pi$.

Ex. Consider a point at $(0,-R)$ on a circle of radius R that is rotating clockwise.



(0, -R)

$$x = R\cos\left(\frac{3\pi}{2} - t\right)$$

$$y = R\sin\left(\frac{3\pi}{2} - t\right)$$

or:

$$x = -R\sin t \qquad y = -R\cos t$$

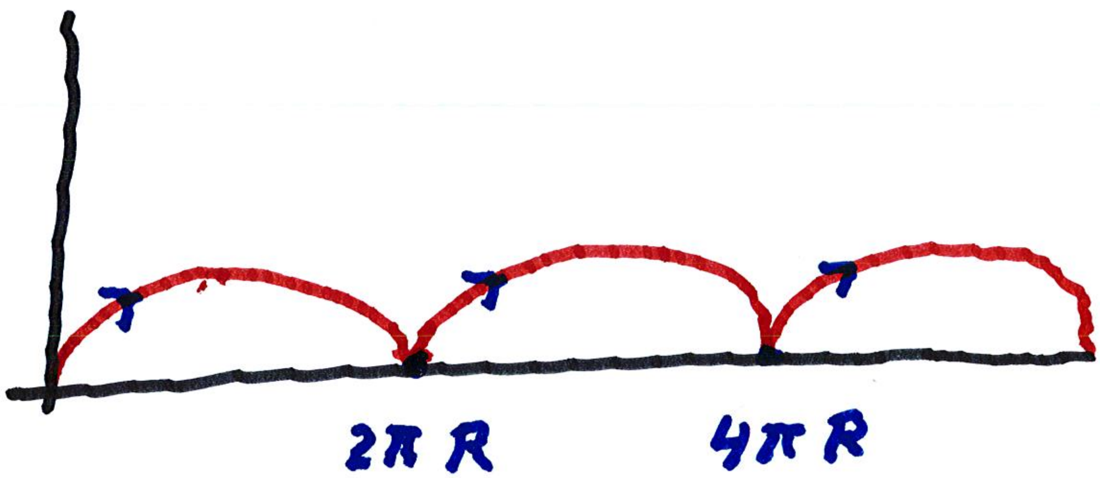
Now lift it up so the circle has a center = (0, R)

$$x = -R\sin t\,, \qquad y = R - R\cos t$$

Now suppose the circle is allowed to roll along the x-axis. After t seconds the particle will move to the right a distance of Rt units. The particle will be at

$$x = Rt - R\sin t \,, \quad y = R - R\cos t$$

The particle follows this path.

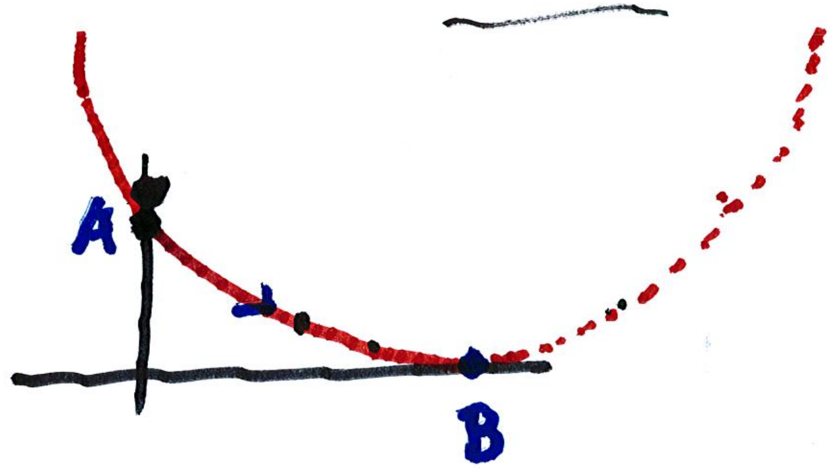


This is called a cycloid

It has a cusp at each point where $t = 0$, $t = 2\pi$, $t = 4\pi$, etc.

Problem : Suppose a particle starts at a point A on a curve. C and slides to the bottom at B. Find the curve C for which it takes the minimum time :

The curve C is the <u>inverted</u> cycloid.



Ex.

Find a parameterization

of the hyperbola

$x^2 - y^2 = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Set $x = \cosh t$ $y = \sinh t$

$\cosh^2 t - \sinh^2 t = 1.$

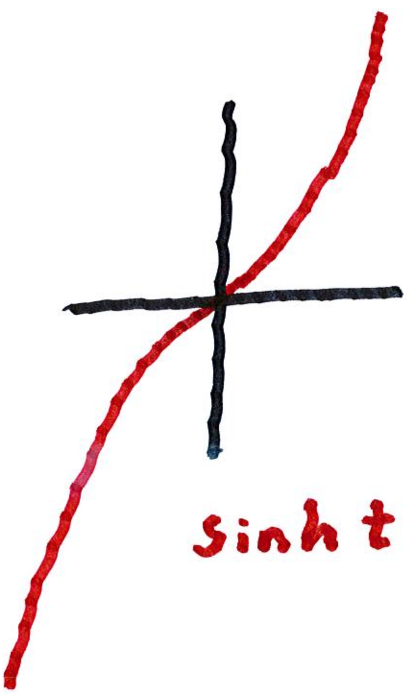
or more generally

$x = a\cosh t$, $y = b\sinh t$

for $-\infty < t < \infty$.

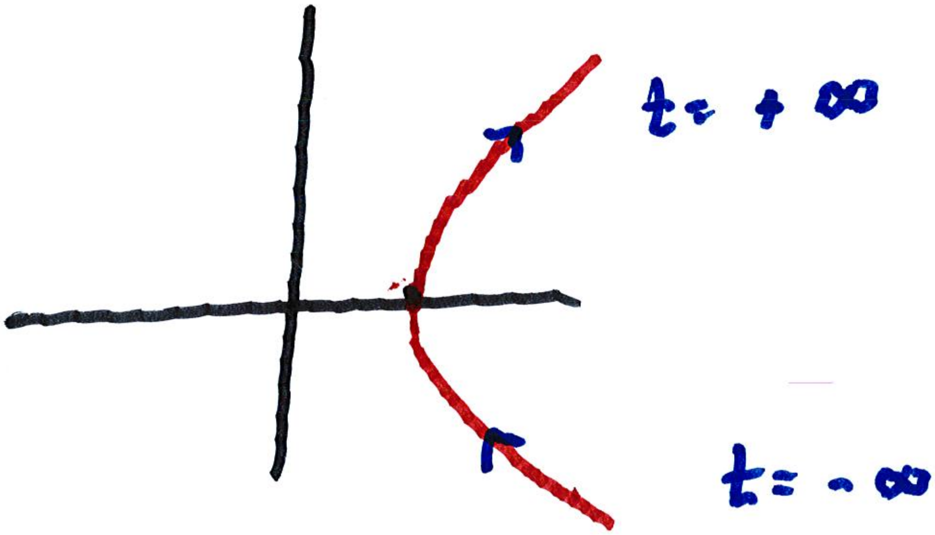
$$\cosh t = \frac{e^{t} + e^{-t}}{2} > 0$$

$$\sinh t = \frac{e^{t} - e^{-t}}{2}$$



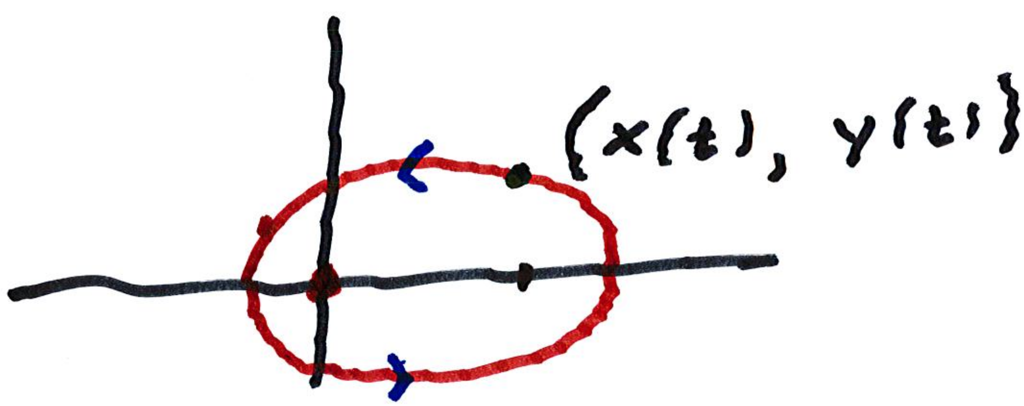
sinh t

cosh t



Kepler's Conjecture.

Given a planet in the influence of the inward-pointing gravitational force. Then the planet travels in an elliptical path, where the sun is at one of the foci



It is known that we cannot solve for the parameterization of the planet.

Ex. Describe the path if

$$x = \sqrt{t+1} \quad , \quad y = \sqrt{t-1}$$

$(\text{if } t \geq 1)$

$x^2 = t+1 \qquad\qquad y^2 = t-1$

$\rightarrow t = x^2 - 1 \qquad\qquad \rightarrow t = y^2 + 1$

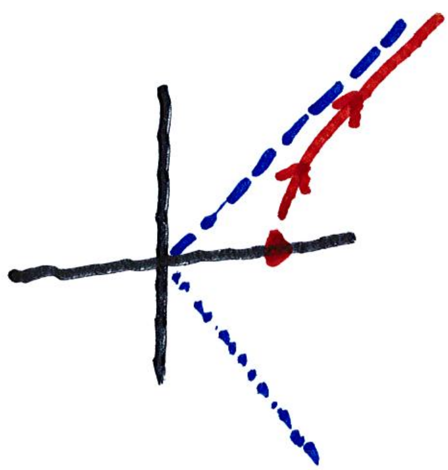
$\therefore \; y^2 + 1 = x^2 - 1$

$\rightarrow x^2 - y^2 = 2$ a hyperbola

When $t=1$, $y=0$

Also, $y \to \infty$ as $t \to$

and when $t \geq 1$, $x \to \infty$



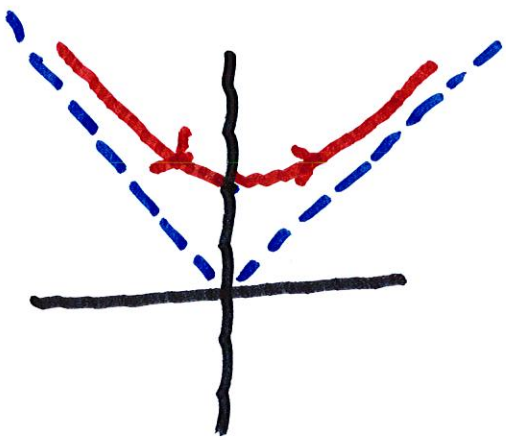
Ex. $x = \sinh t$, $y = \cosh t$

$$\cosh^2 t - \sinh^2 t = 1$$

$$\to \quad y^2 - x^2 = 1$$

Recall as t goes from $-\infty$ to ∞.

$\sinh t$ also goes from $-\infty$ to ∞



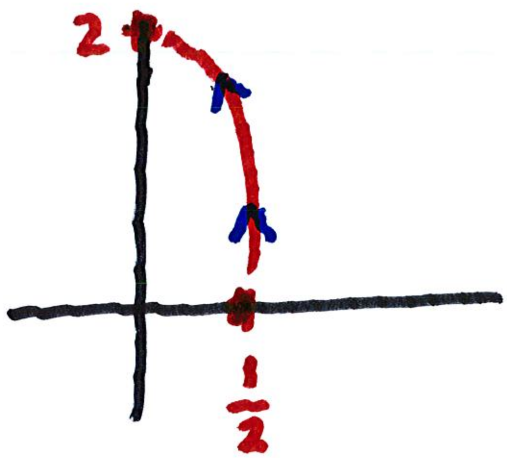
Ex. Find the curve if

$$x = \frac{1}{2}\cos t \quad , \quad y = 2\sin t \; , \quad 0 < t < \frac{\pi}{2}$$

$$4x^2 = \cos^2 t \qquad \frac{y^2}{4} = \sin^2 t$$

$$\rightarrow \quad 4x^2 + \frac{y^2}{4} = 1 \qquad \frac{x^2}{\left(\frac{1}{4}\right)} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{2^2} = 1$$



x, y both > 0

Ex. Find the curve if

$x = \tan^2\theta \qquad y = \sec\theta \,, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sec^2\theta = 1 + \tan^2\theta$

$\rightarrow \quad y^2 = 1 + x$

and $x > 0$,
$y > 0$