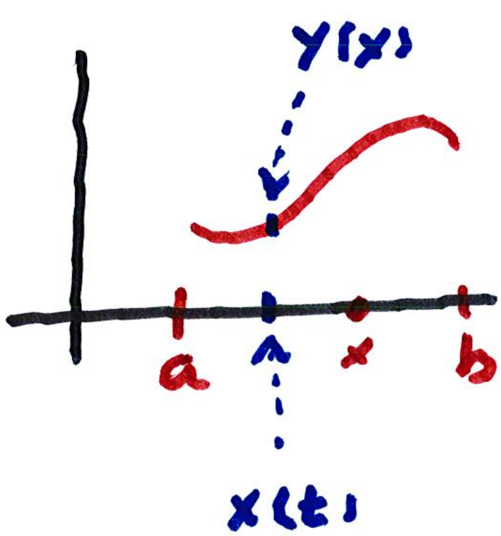
10.2 Calculus with Param. Curves

Consider a parametric curve defined by

$$x = x(t) \quad \text{and} \quad y = y(t).$$

If $\dfrac{dx}{dt} \overset{>0}{\neq 0}$ on an interval (a,b), then this defines a function $y = y(x)$.



we can define a fcn. $y(x)$ so

$$y(t) = y(x(t))$$

We can differentiate this

curve:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{(Chain Rule)}$$

$$\rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

If we replace y by $\frac{dy}{dx}$,

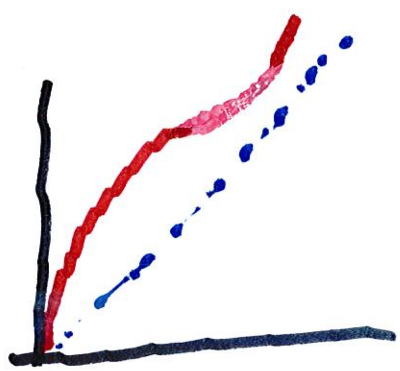
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Consider the curve

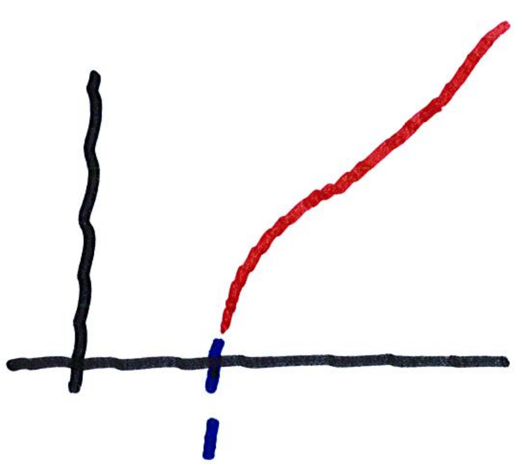
$x = t^2 \qquad y = t^2 + t \qquad t > 0$

Solve for t: $\quad t = \sqrt{x}$

$\rightarrow \quad y = x + \sqrt{x}$



Shift x by 1 $\qquad$ Use $x = t^2 + 1$



$$\frac{dy}{dx} = \frac{2t+1}{2t} = 1 + \frac{1}{2t} = \underline{\underline{\frac{2t+1}{2t}}}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(1 - \frac{1}{2t}\right)}{2t}$$

$$= -\frac{1}{2t^2} + \frac{\frac{1}{2t^2}}{2t}$$

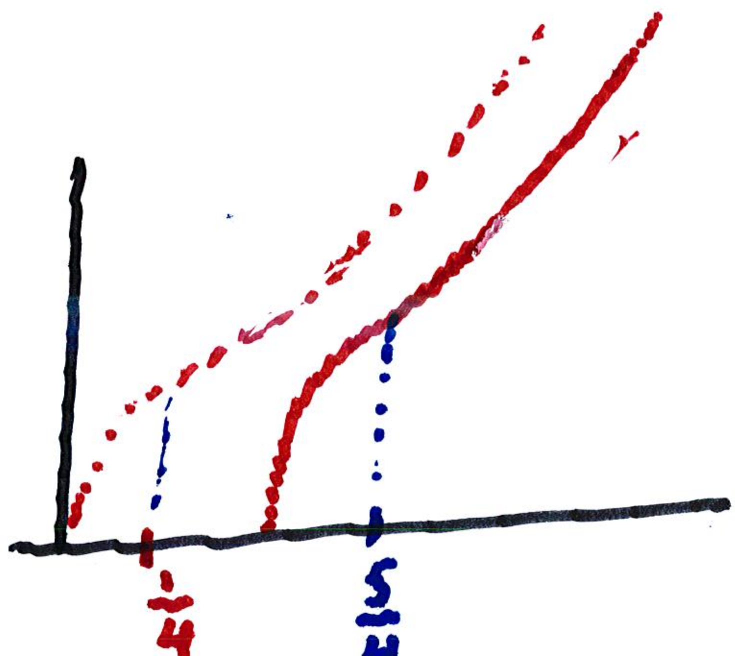
$$= - \frac{-2t+1}{4t^3}$$

$$= \frac{-1+2t}{4t^3}$$

$\therefore \dfrac{dy}{dx} > 0$ if $t > 0$

Also, $\dfrac{d^2y}{dx^2} > 0$ if $t > \frac{1}{2}$

and < 0 if $t < \frac{1}{2}$



$t = \frac{1}{2} \rightarrow \quad x = \frac{5}{4}$

Note that $\frac{dy}{dx} = \infty$ as $t \to 0^+$

and $\frac{dy}{dx}$ for all $t > 0$.

Ex. Find all points where the curve has either a horizontal or vertical tangent.

$x = t^3 - 3t \quad , \quad y = t^2 - 3$

hor. tan

if $\frac{dy}{dt} = 0 \to t = 0$

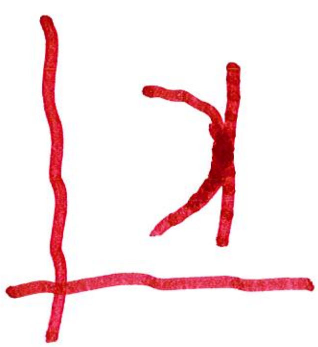
$$\frac{dy}{dx} = \frac{\overset{\frac{dy}{dt}}{2t}}{3t^2-3} = 0 \text{ if } t = 0$$

~~dx~~ $\frac{dx}{dt}$

Also $\frac{dy}{dx} = \infty$ if $t = \pm 1$

$(0, -3)$

$(-2, -2)$ and $(2, -2)$



vert. tan

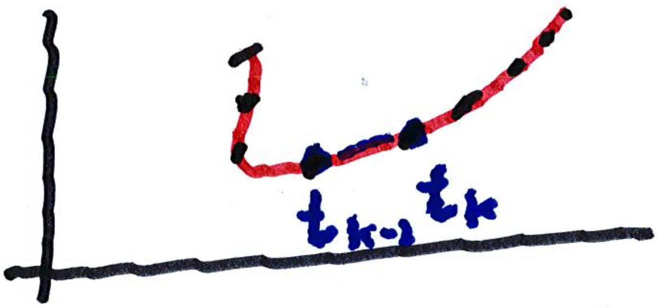
look for t

where $\frac{dx}{dt} = 0$

vert. tan if $\frac{dx}{dt} = 0$

Consider the curve

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$



$$a = t_0 < t_1 < \dots < t_{k-1} < t_k < \dots < t_N = b$$

$$(x_{t_{k-1}}, y_{t_k}) \text{ to } (x_{t_k}, y_{t_k})$$

Length of segment

is $\sqrt{\left(x(t_k)-x(t_{k-1})\right)^2 + \left(y(t_k)-y(t_{k-1})\right)^2}$

$$\Delta L \approx \sqrt{(x'(t_k))^2(\Delta t)^2 + (y'(t_k))^2(\Delta t)^2}$$

$$= \sqrt{(x'(t_k))^2 + (y'(t_k))^2}\quad \Delta t$$

Now add up for all $k = 1, 2, \ldots, N$,

$$L \approx \sum_{k=1}^{N} \sqrt{(x'(t_k))^2 + (y'(t_k))^2}\, \Delta t$$

Let $N \to \infty$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex. Find length of

$x = e^t + e^{-t}$, $\quad y = 5 - 2t$,

$0 \le t \le 3$.

$$\frac{dx}{dt} = e^{t} - e^{-t} \qquad \frac{dy}{dt} = -2$$

$$\left(\frac{dx}{dt}\right)^2 = e^{2t} - 2 + e^{-2t}$$

$$\left(\frac{dy}{dt}\right)^2 = 4$$

$$\therefore L = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\ dt$$

$$= \int_0^3 \sqrt{(e^{t} + e^{-t})^2}\ dt$$

$$= \int_0^3 e^t + e^{-t} \, dt$$

$$= e^t - e^{-t} \Big|_0^3 = \underline{\underline{e^3 - e^{-3}}}$$

Find length and

Ex. Graph ~~curve~~ of (1)

$x = e^t \cos t \qquad y = e^t \sin t,$

$0 \le t \le \pi$

$\frac{dx}{dt} = e^t \cos t - e^t \sin t = 0 \, ?$

$\cos t - \sin t = 0$

$\frac{dy}{dt} = e^t \sin t + e^t \cos t$

$\frac{dy}{dt} = 0 \, ? \qquad \sin t + \cos t = 0$

$$= \sqrt{e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right.}$$

$$= \sqrt{2e^{2t}} = \sqrt{2}\,e^{t}$$

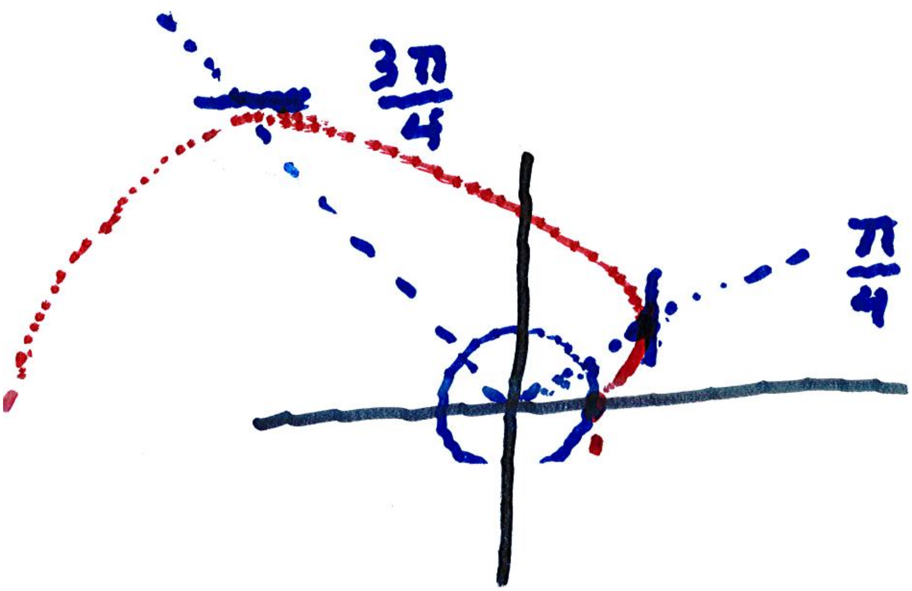
$$\therefore L = \int_0^{\pi} \sqrt{2}\, e^{t}\, dt = \sqrt{2}\left(e^{\pi} - 1\right)$$

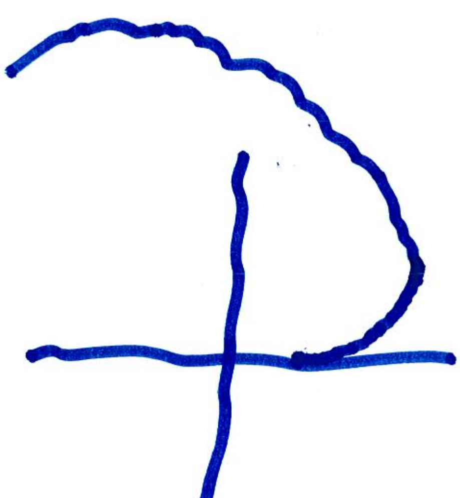
Sketch curve in (1).

$$\frac{dx}{dt} = 0 \quad \text{if} \quad \cos t = \sin t$$

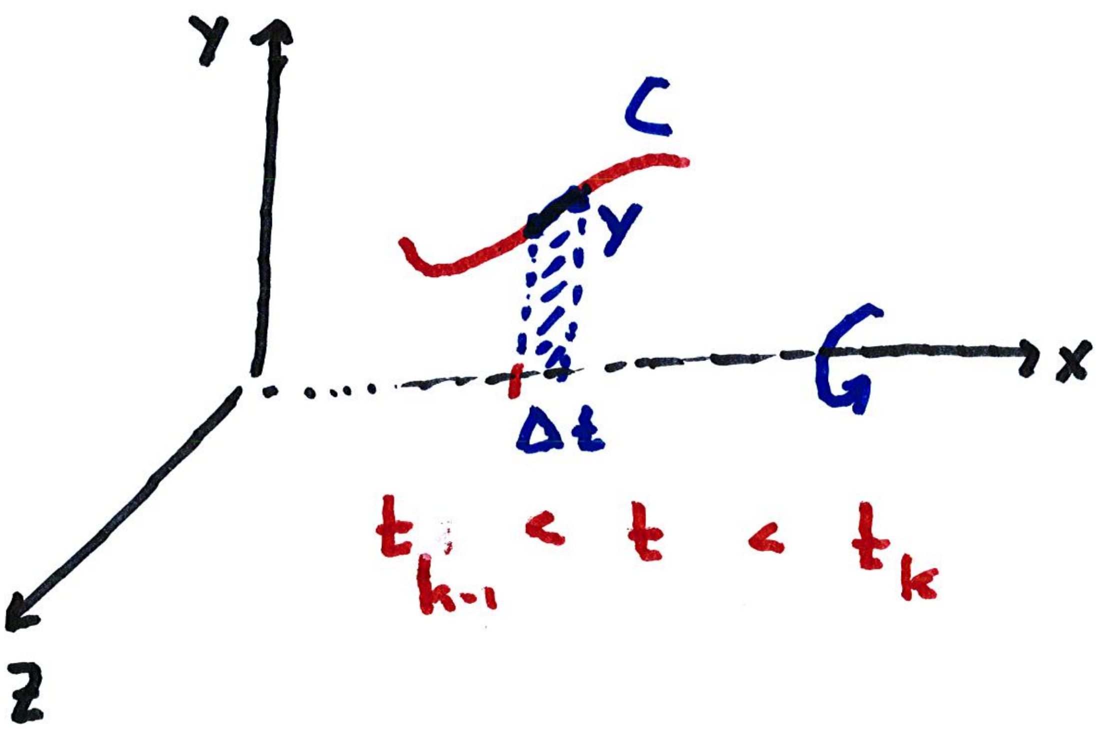
$$t = \frac{\pi}{4}$$

$$\frac{dy}{dt} = 0 \quad \text{if} \quad \sin t = -\cos t$$





Consider a curve C defined by $x = x(t)$ and $y = y(t)$ for $a \le t \le b$. Find the surface area of the surface obtained by rotating C about the x-axis



$$dS = 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\therefore S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Let C = the curve

$x(t) = t\sin t$, $y(t) = t\cos t$,

$0 \le t \le \frac{\pi}{2}$

What is the surface area if we rotate C about x-axis?

5A :

$$\int_0^{\frac{\pi}{2}} 2\pi t \cos t \cdot \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\, dt$$

Don't have to solve.

Ex. At what points on the curve

$x = 2t^3$, $y = 1 + 4t - t^2$ does

the tangent line have slope 1

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4-2t}{6t^2} = \frac{2-t}{3t^2} = 1$$

$$\therefore\ 3t^2 = 2-t$$

$$\text{or}\quad 3t^2 + t - 2 = 0$$

$$\rightarrow t = \frac{-1 \pm \sqrt{1+48}}{6}$$

$$\therefore\ t = -\frac{4}{3} \quad \text{or} \quad t = +1$$

$$\therefore \qquad x = \frac{128}{27}, \quad y = 1 + \frac{16}{3} - \frac{16}{9}$$

$$= \frac{41}{9}$$

$$\left(\frac{128}{27}, \frac{41}{9} \right)$$

or: $(-2, -4)$

$x = t^2 + 1, \quad y = t^2 + t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(1 + \frac{1}{2t}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{2t^2}}{2t}$$

$$= -\frac{1}{4t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4t^2} < 0$$

(if $t \neq 0$)