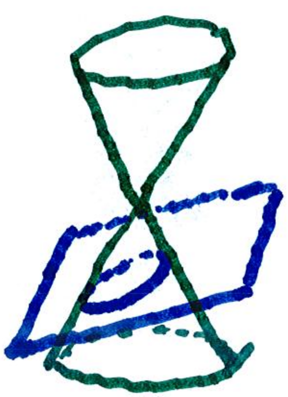
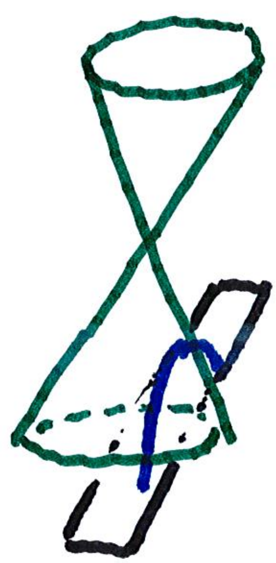
10.5 Conic Sections

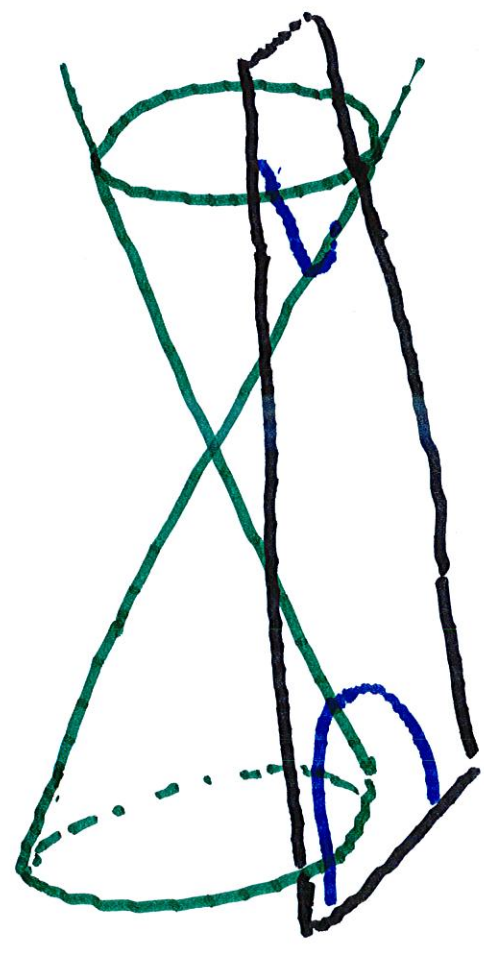
These curves come about by taking the intersection of a plane with a cone.



ellipse



parabola



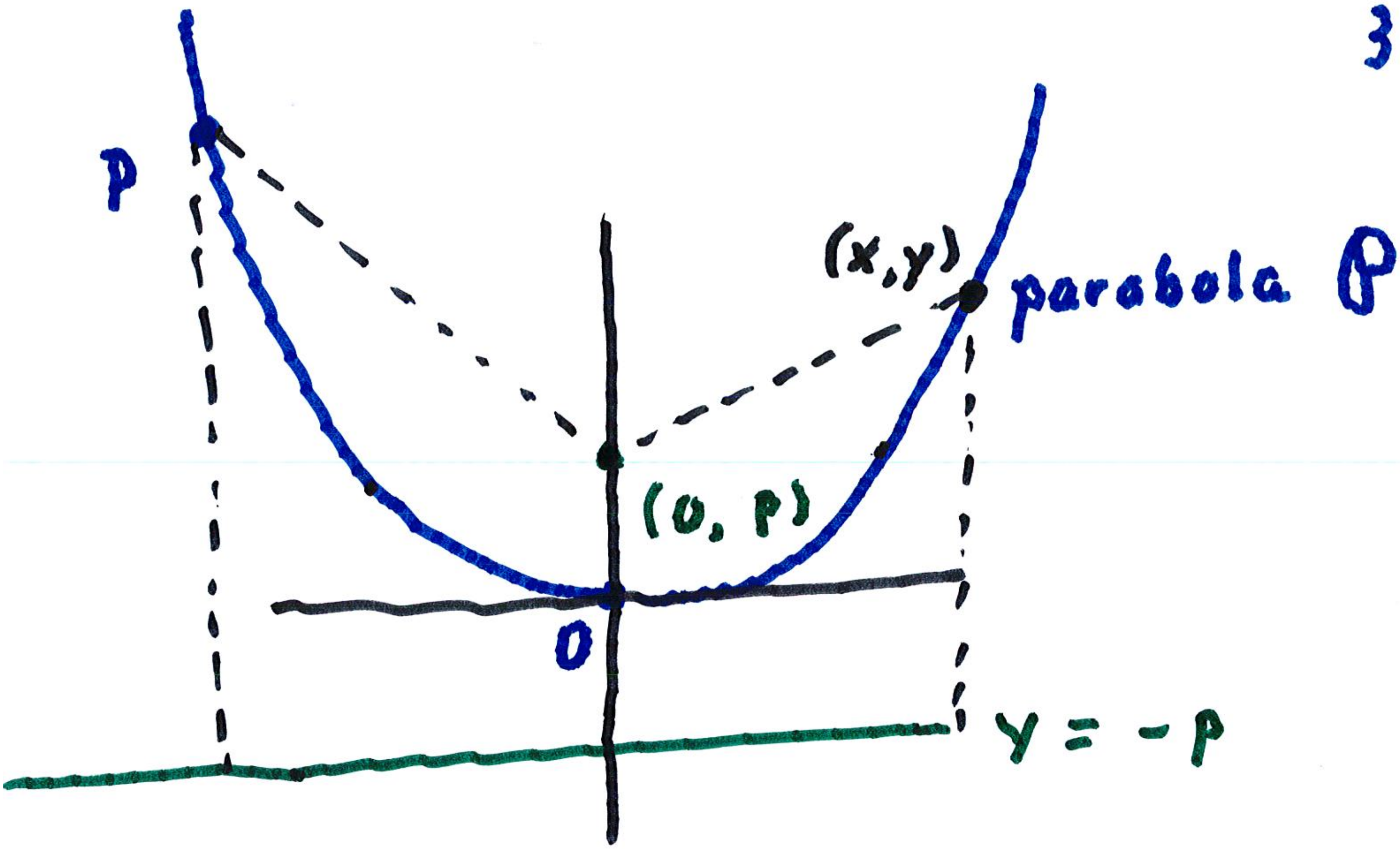
hyperbola

Consider a line L called the directrix and a point F called the focus (not in L).

A parabola is the set of points P such that

$$d(P,F) = d(P,L)$$

We can describe L as the line $y = -p$ and F as the point $(0, p)$



Suppose $P(x,y)$ is in the

parabola $\mathcal{P}$.

$$y + p = \sqrt{x^2 + (y-p)^2}$$

$$y^2 + 2py + p^2 = x^2 + y^2 - 2py + p^2$$

Simplify

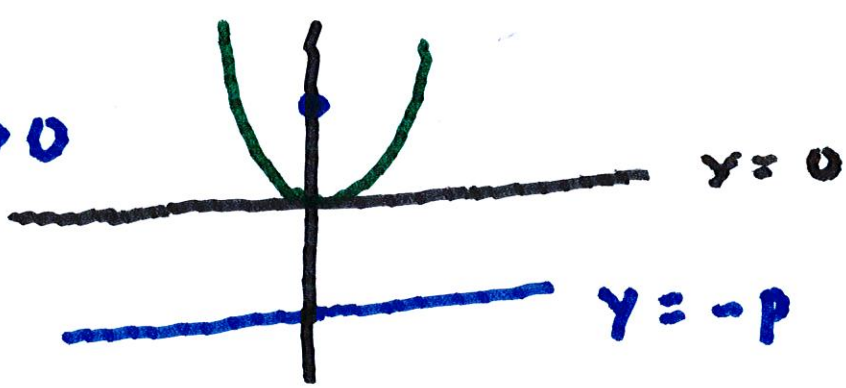
$$4py = x^2.$$

$\therefore$ If the focus is $(0, p)$

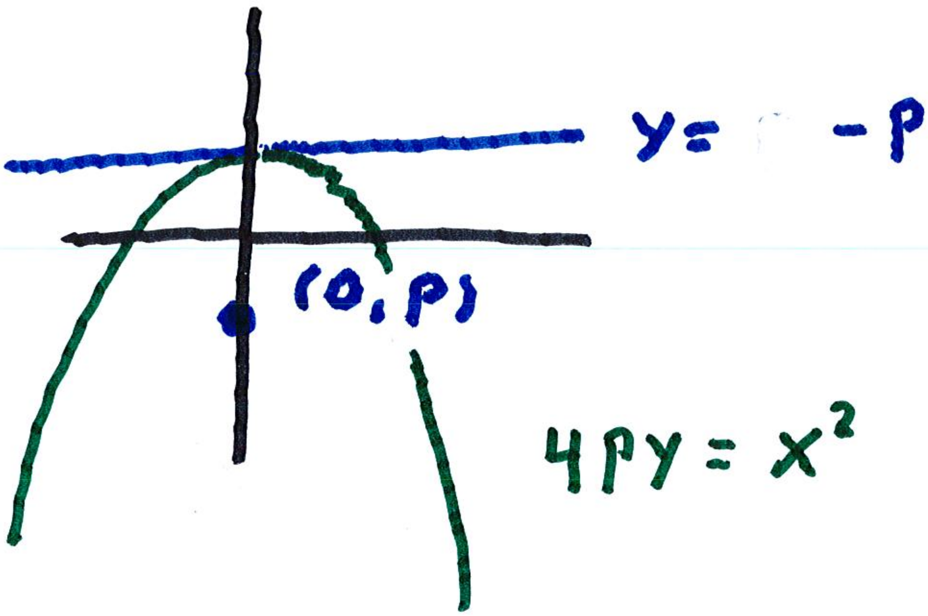
and the directrix is $y = -p$,

then $\wp$ is $4py = x^2$

$p > 0$



If $p<0$, then the parabola is



Similarly if the focus F is $(p, 0)$

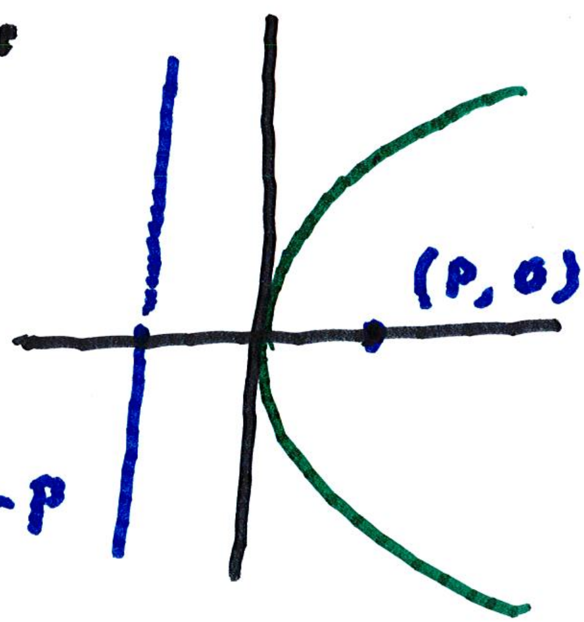
$(p > 0)$

and the directrix is

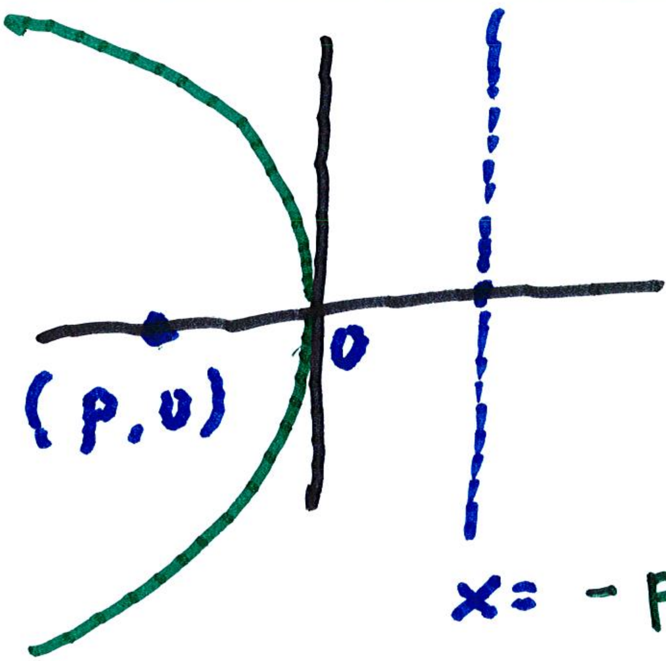
$x = -p$, then the

parabola is

$4px = y^2$



If $p < 0$ and the focus is $(p, 0)$, then the parabola is



Ex. Find the focus and the directrix of $x^2 + 2y = 0$

$\rightarrow x^2 = -2y$ Hence $4p = -2$

$x^2 = 4py$ $\rightarrow p = -\frac{1}{2}$

$\therefore$ focus is at $\left(0, -\frac{1}{2}\right)$

and the directrix is

$y = -p$ or $y = -\left(-\frac{1}{2}\right)$

or $y = \frac{1}{2}$

Ellipses Consider 2 points

F_1 and F_2 (called the foci)

we let $F_1 = (-c, 0)$ and

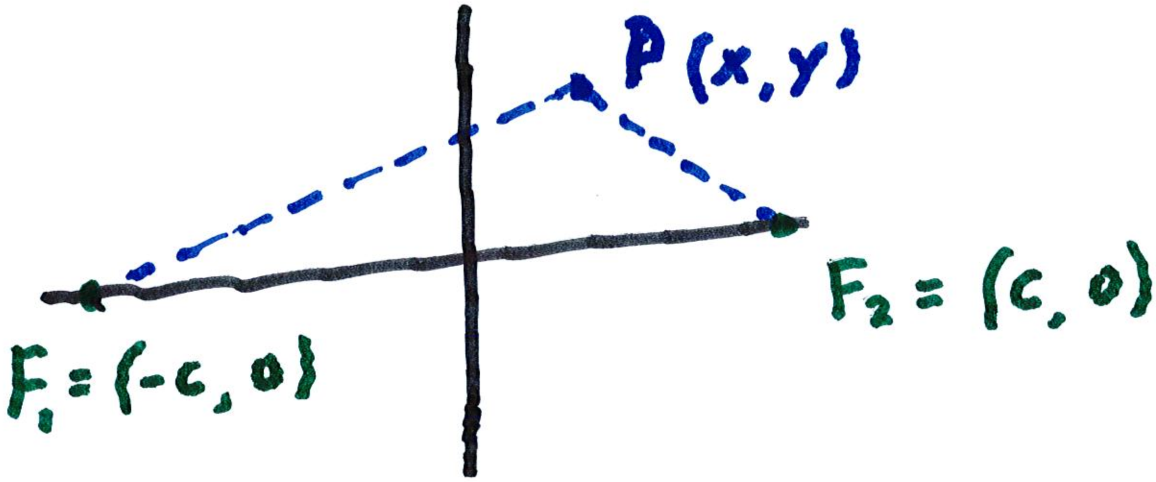
$F_2 = (c, 0)$.

Note the distance between F_1 and F_2 is $= 2c$. Let a be a number with $2a > 2c$

We let the ellipse E be the set of points P such that

$$\text{dist}(F_1, P) + \text{dist}(F_2, P) = 2a.$$

If we set $a = c$, then the origin O would be in the ellipse.



$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Square both sides

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}$$
$$+(x-c)^2+y^2$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

Divide by 4 and square:

$$(cx - a^2)^2 = a^2\left((x-c)^2 + y^2\right)$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$a^4 - a^2c^2 = (a^2 - c^2)x^2 + a^2y^2$$

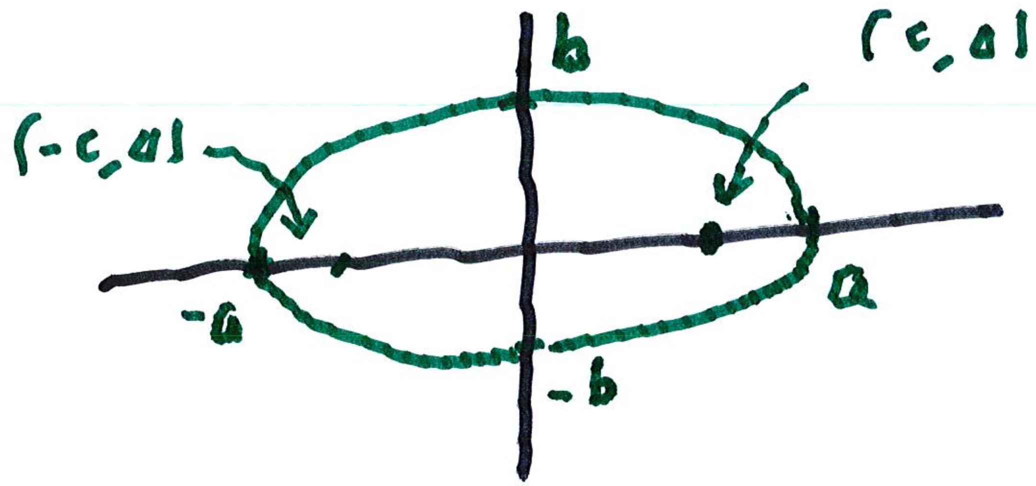
Set $b^2 = a^2 - c^2$.

$$a^2b^2 = b^2x^2 + a^2y^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2$, it follows

that $a > b$



Let $(-a, 0)$ and $(a, 0)$ be

the vertices of the ellipse.

The segment from $(-a, 0)$

to $(a, 0)$ be the major axis,

and from $(0, -b)$ to $(0, b)$ the

minor axis.

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

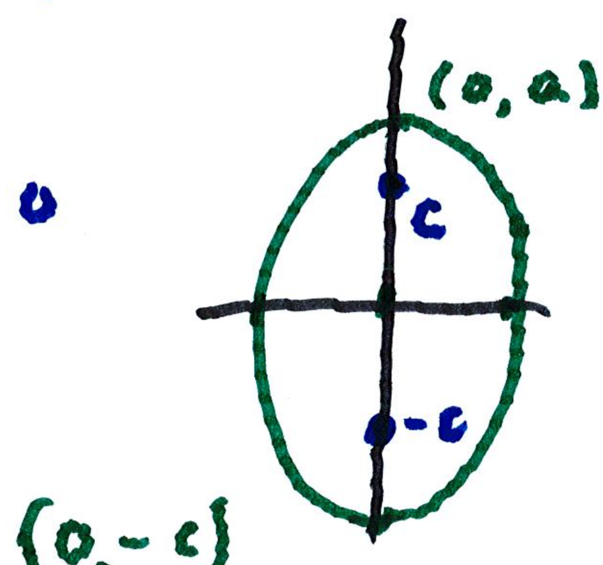
with $a \geq b > 0$ has foci

at $(\pm c, 0)$ where $c^2 = a^2 - b^2$

The ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

has foci at $(0, \pm c)$

where $a \geq b > 0$



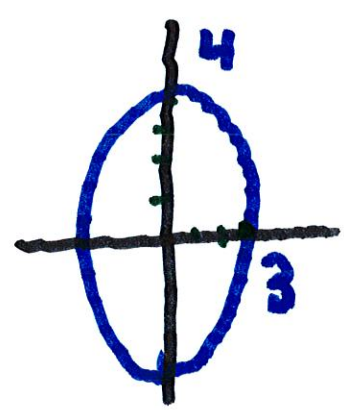
Ex. Find the equation of the ellipse with foci at $(0, \pm\sqrt{7})$ and vertices at $(0, \pm 4)$

Note $a = 4$

$c^2 = a^2 - b^2$ $\qquad c = \sqrt{7} \rightarrow c^2 = 7$

$\therefore 7 = 16 - b^2 \rightarrow b = 3 \rightarrow b^2 = 9$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$



Hyperbolas

Suppose F_1 and F_2 are foci

with $F_1 = (-c, 0)$ and $F_2 = (c, 0)$

Let $\underline{a}$ be any number with

$0 < a < c$

We define a hyperbola H

to be the set of points P

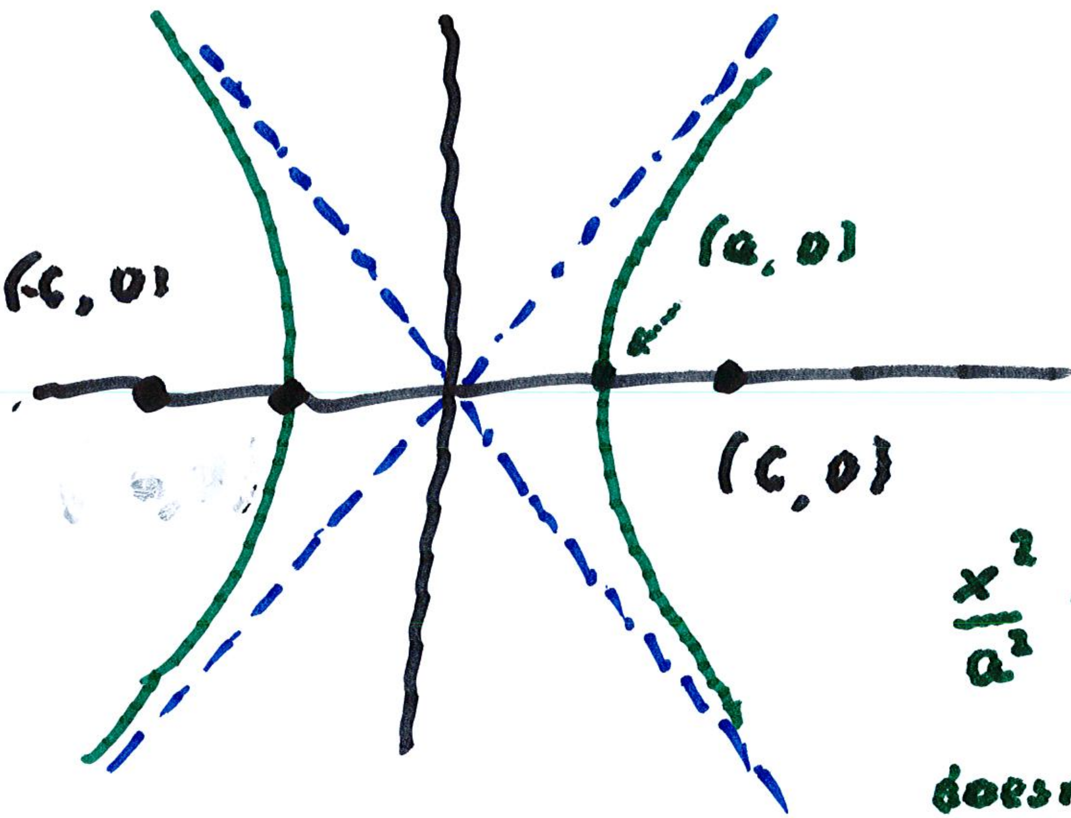
such that $|PF_2| - |PF_1| = 2a$

Setting $b^2 = c^2 - a^2$.

and following the same method as for ellipses. we obtained

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The hyperbola has foci at $(\pm c, 0)$ and vertices at $(\pm a, 0)$. The hyperbola has asymptotes at $y = \frac{\pm bx}{a}$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

doesn't cross y-axis

For the equation

$\frac{y^2}{4} - \frac{x^2}{1} = 1$, we always

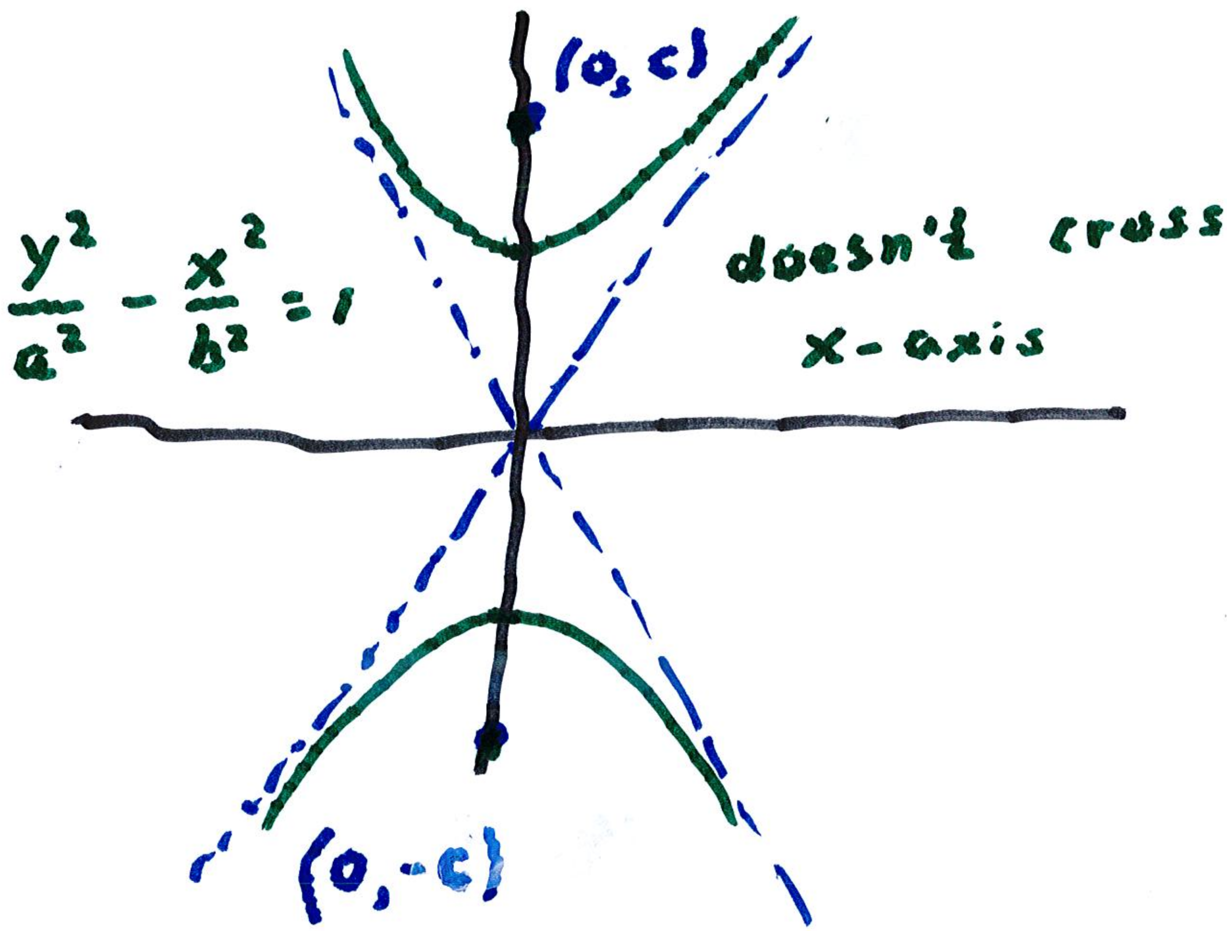
define a^2 = denominator

of the positive term

$a^2 = 4$
$b^2 = 1$

Also $a^2 + b^2 = c^2$.

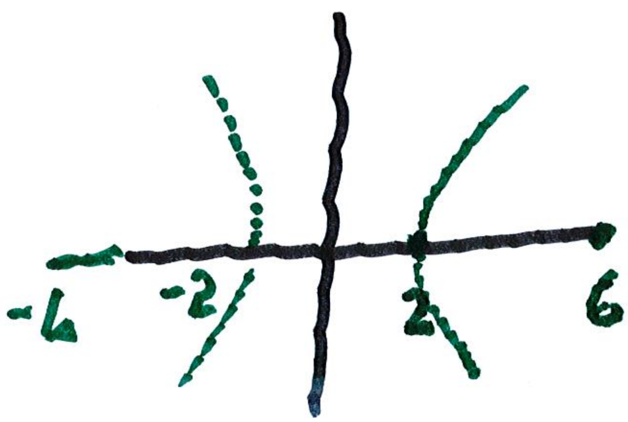
so $4 + 1 = c^2 \to c = \sqrt{5}$.



Ex. Suppose vertices at

$(\pm 2, 0)$ and foci at $(\pm 6, 0)$

Find equation of hyperbola



$a^2 + b^2 = c^2 = 36$

$a^2 = 4$

$\to b^2 = 36 - 4 = 32$

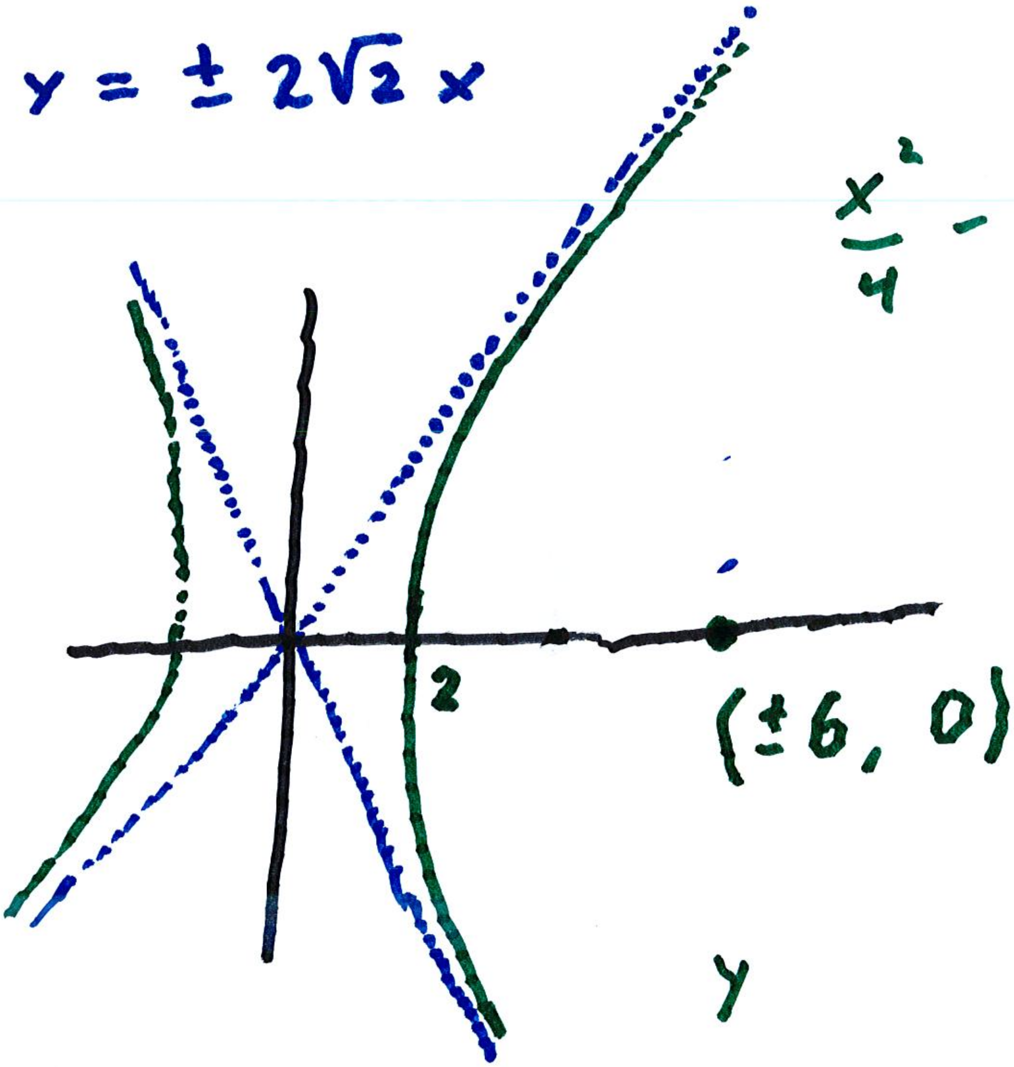
$\therefore\ b = 4\sqrt{2}$

$$\frac{x^2}{4} - \frac{y^2}{32} = 1$$

asym. at $\dfrac{y^2}{32} = \dfrac{x^2}{4}$ $\quad y^2 = 8x^2$

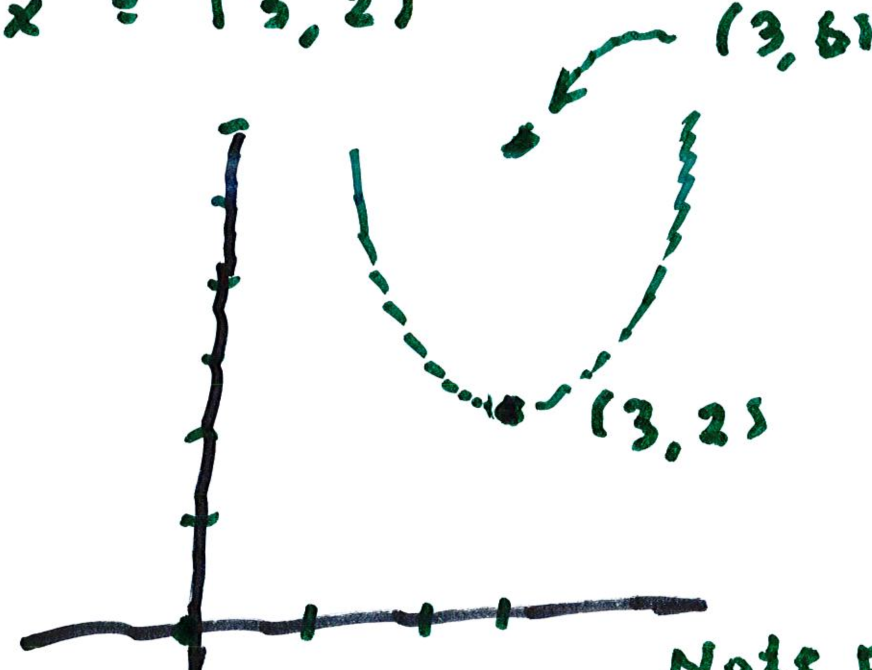
$\therefore y = \pm 2\sqrt{2}\, x$

$\frac{x^2}{4} - \frac{y^2}{32} = 0$



$$\frac{(x-2)^3}{9} + \frac{(y-1)^2}{8} = 1$$

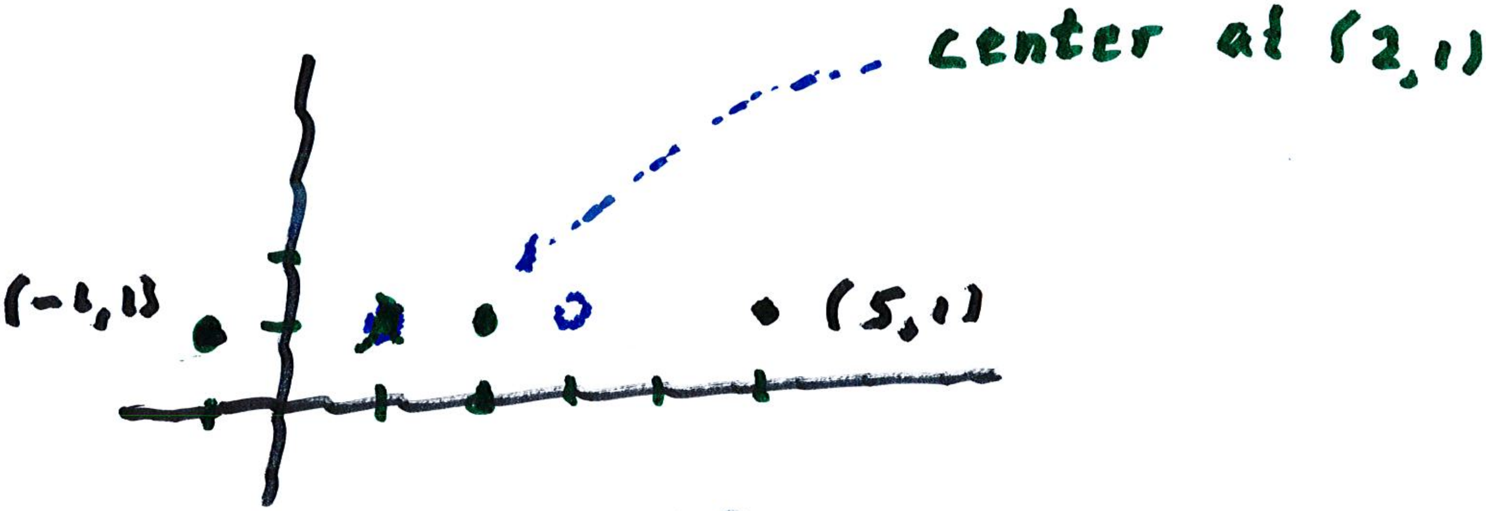
Ex. Find eq'n. of parabola with focus = (3, 6) and vertex = (3, 2)



Note $p = 4$

$$4p(y-2) = (x-3)^2$$

Ex. Find the equation of the ellipse with vertices at $(-1,1)$, $(5,1)$ and foci at $(1,1)$, $(3,1)$



$$\left.\begin{array}{l} a = 3 \\ c = 1 \end{array}\right\} \quad b^2 = a^2 - c^2$$

$$b^2 = 9 - 1 = 8$$

$$b = 2\sqrt{2}$$

Note $p = 6 - 2 = 4$

$\therefore \quad 16(y-2) = (x-3)^2$