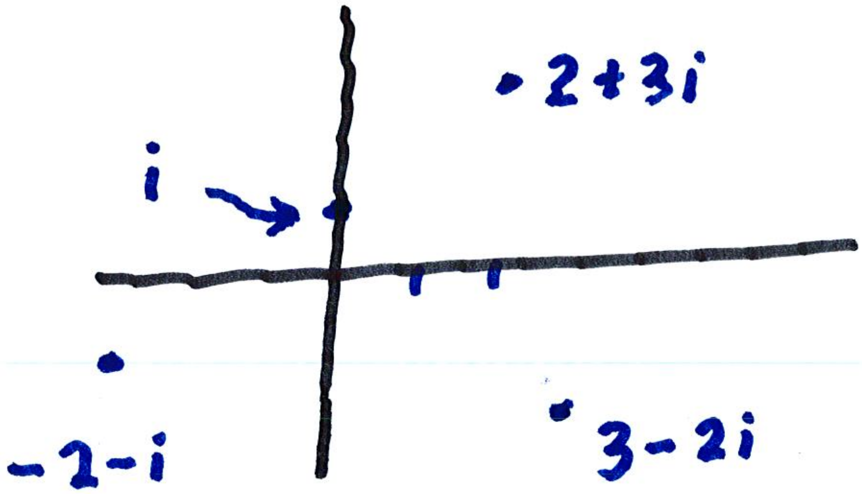
Complex Numbers (Appendix H)

A complex number is an expression of the form $z = a + bi$, where i is a symbol of the form $i^2 = -1$.

A complex number $a+bi$ can be represented by the ordered pair (a, b). The number i is identified with $(0,1)$



We define

$$z + w = (a+bi) + (c+di)$$

$$= (a+c) + (b+di)$$

$$\therefore \ (2+3i) + (4-2i)$$

$$= 6 + i$$

For multiplication, we use $i^2 = -1$

$(a+bi)\cdot(c+di)$

$= (ac - bd) + (ad + bc)i$

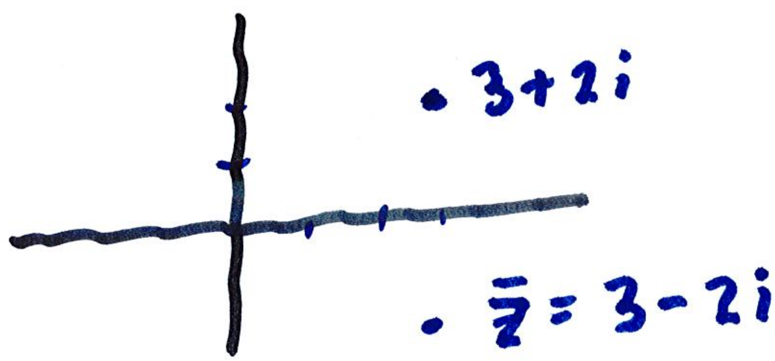
$\therefore (3+2i)(2-i)$

$= (3\cdot 2 + 2) + (-3 + 4)i$

$= \underline{8 + i}$

For division, it is useful to define the complex conjugate

For $z = a + bi$, we define the complex conjugate of z by $\bar{z} = a - bi$.



Ex. Express $\dfrac{-2+3i}{3+5i}$ in the form $\underline{a+bi}$

We multiply numerator and denominator by the complex conjugate of $3+5i$:

$$\frac{-2+3i}{3+5i} \cdot \frac{3-5i}{3-5i}$$

$$= \frac{(-6+15)+(10+9)i}{3^2+5^2}$$

$$= \frac{9}{34} + \frac{19}{34}i$$

If $z = a+bi$, we define

$$|z| = \sqrt{a^2+b^2}.$$

Note that

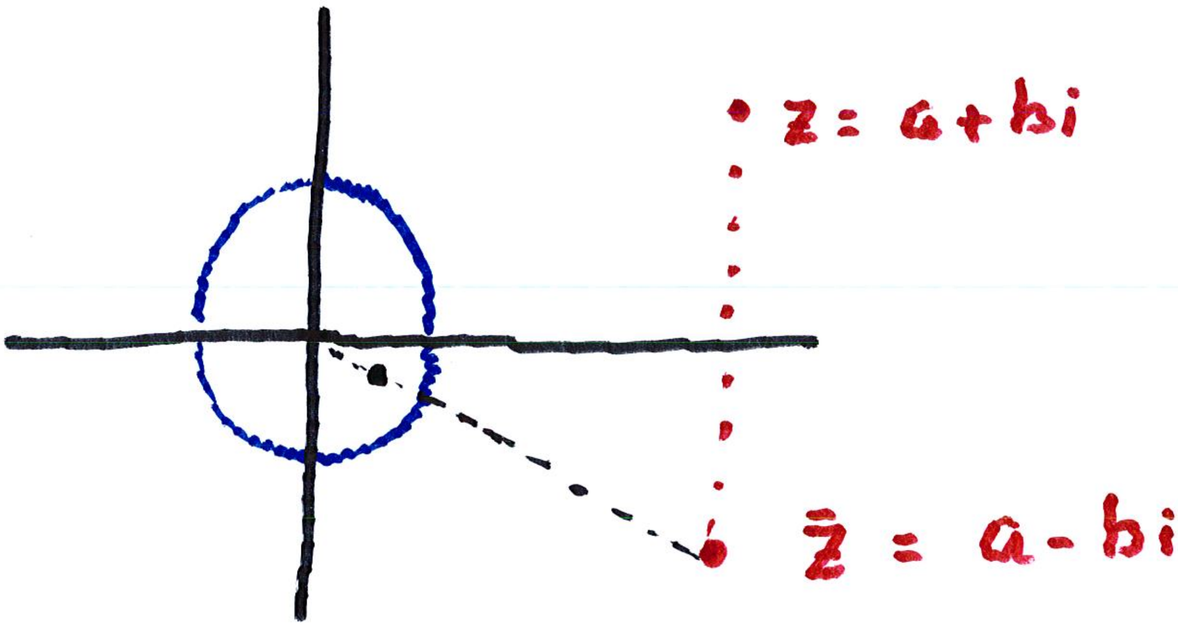
$$z\bar{z} = (a+bi)(a-bi)$$

$$= a^2 + abi - abi + b^2$$

$$= a^2 + b^2$$

$$\therefore \quad z\bar{z} = |z|^2.$$

or $$z\left(\frac{\bar{z}}{|z|^2}\right) = 1$$



If $c > 0$, then $(\sqrt{c}\, i)^2 = ci^2 = -c$

Also $(-\sqrt{c}\, i)^2 = ci^2 = -c$.

Hence we can write

$$\sqrt{-c} = \pm\sqrt{c}\, i$$

This makes it possibe to solve quadratic equations:

Solve $x^2 + 4x + 17 = 0$

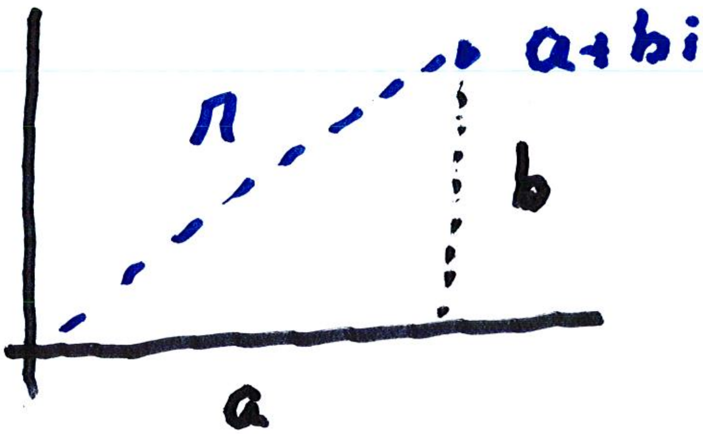
$$(x^2 + 4x + 4) = -17 + 4 = -13$$

$$\therefore \quad (x+2)^2 = -13$$

$$x + 2 = \pm\sqrt{13}\, i$$

$$x = -2 \pm \sqrt{13}\, i$$

Polar form



$$\frac{a}{r} = \cos\theta \qquad \frac{b}{r} = \sin\theta$$

$$a = r\cos\theta \qquad b = r\sin\theta$$

$$\therefore \quad z = a+bi = r\cos\theta + r\sin\theta\, i$$

$$z = r(\cos\theta + i\sin\theta)$$

This is the polar form of z.

Ex. Express $z = 2 + 2\sqrt{3}\, i$

in polar form:

$$r = \sqrt{2^2 + (2\sqrt{3})^2}$$

$$= \sqrt{4 + 12} = 4.$$

Also $\tan\theta = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

Hence, $\theta = \dfrac{\pi}{3}$

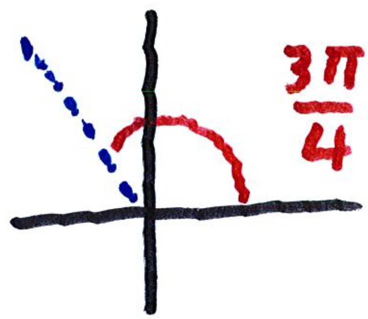
$$\rightarrow \quad z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

polar form

Ex. If $z = -3 + 3i$,

$$r = \sqrt{(-3)^2 + (3)^2} = \sqrt{18}$$

$$r = 3\sqrt{2}$$



Also, $\theta = \frac{3\pi}{4}$,

$$\therefore z = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Using the polar form, we can explain complex multiplication.

Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2\Big((\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)\Big)$$

$$= r_1 r_2 \Big((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$$

$$+ i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \Big)$$

$$= r_1 r_2 \big(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \big)$$

To multiply 2 complex numbers, multiply the absolute values, and add the angles.

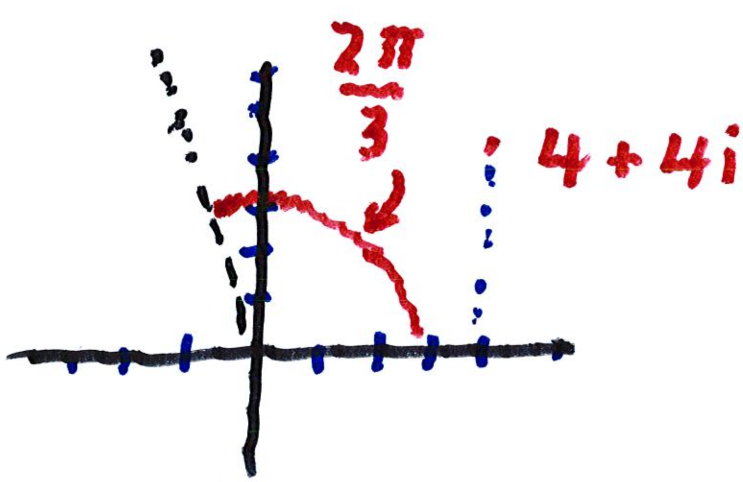
Ex. Let $z_1 = 4 + 4i$,

and $z_2 = -3 + 3\sqrt{3}\, i$.

$$r_1 = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

and $r_2 = \sqrt{(-3)^2 + (3\sqrt{3})^2}$

$$= \sqrt{9 + 27} = 6$$



$$\theta_1 = \frac{\pi}{4}$$

and $\theta_2 = \dfrac{2\pi}{3}$

$$\therefore \quad r_1 r_2 = 4\sqrt{2} \cdot 6 = 24\sqrt{2}$$

$$\theta_1 + \theta_2 = \frac{\pi}{4} + \frac{2\pi}{3}$$

$$= \frac{3\pi + 8\pi}{12}$$

$$\therefore Z_1 Z_2 = 24\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

Express $\dfrac{Z_2}{Z_1}$ in polar form

$$= \frac{6}{4\sqrt{2}}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

Recall $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

This converges even if z is a complex number.

One can show $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

We can set $z = iy$. We get

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!}$$

$$= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right)}_{\cos y} + i\underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)}_{\sin y}$$

We get

$$e^{iy} = \cos y + i \sin y$$

and also, if $z = x + iy$

$$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$$

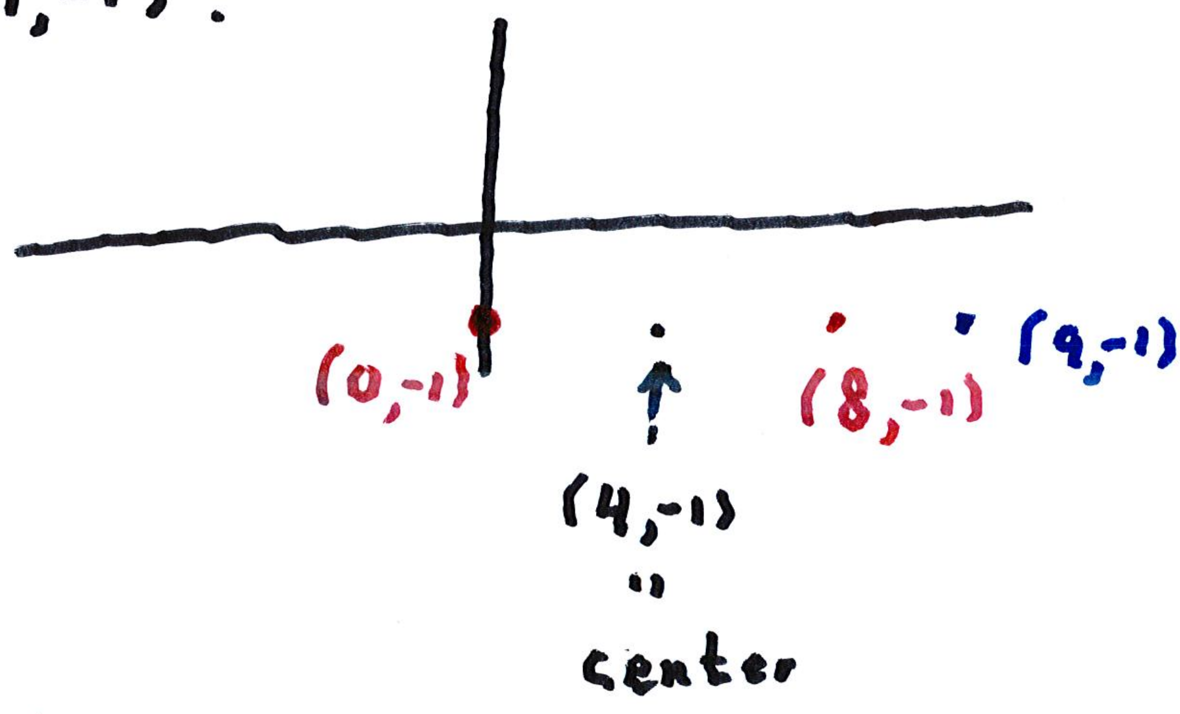
$$= e^{x}(\cos y + i \sin y)$$

Hence,

$$e^{x+iy} = e^{x}(\cos y + i \sin y)$$

for any numbers x and y.

Ex. Find the equation of an ellipse with foci at $(0,-1)$ and $(8,-1)$, and a vertex at $(9,-1)$.



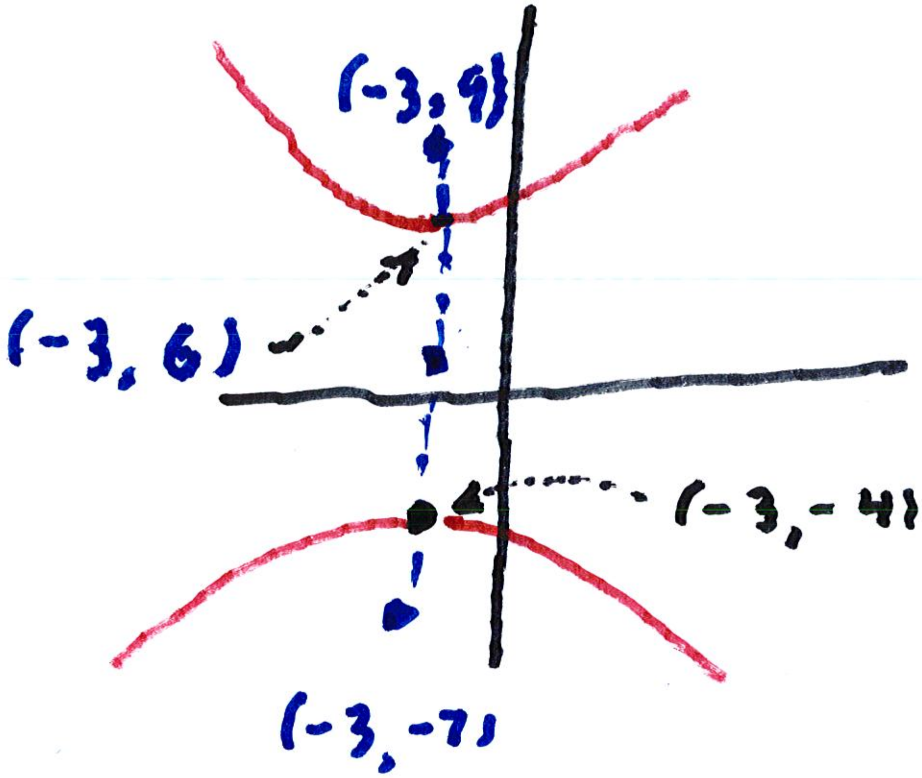
$c = 4$

$a = 5$

$b^2 = a^2 - c^2 \rightarrow b^2 = 25 - 16 = 9$

$$\frac{(x-4)^2}{25} + \frac{(y+1)^2}{9} = 1$$

Ex. Find the equation of the hyperbola with foci at $(-3,-7)$ and $(-3,9)$, and with vertices at $(-3,-4)$ and $(-3,9)$.



center at $(-3, 1)$ $\quad c = 8$

$a^2 = 5^2 \rightarrow 25 + b^2 = 64$

$\therefore \ b^2 = 39$

$$\frac{(y-1)^2}{25} - \frac{(x+3)^2}{39} = 1.$$

Find asymptotes

$$\frac{(y-1)^2}{25} - \frac{(x+3)^2}{39} = 0$$

$$\frac{(y-1)}{5} = \pm \frac{(x+3)}{\cancel{39}\ \sqrt{39}}$$

$$\rightarrow \quad y = 1 \pm \frac{5}{\sqrt{39}}(x+3)$$