12.4 Cross Products.

Suppose $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

We define $\vec{a} \times \vec{b}$ by

$$\vec{a} \times \vec{b} = \left\langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \right\rangle$$

We can also use determinants

2x2 det. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3x3 det.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Compute $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j}$$

$$+ (a_1 b_2 - a_2 b_1)\vec{k}$$

$$= \langle a_2 b_3 - a_3 b_2 , a_3 b_1 - a_1 b_3 , a_1 b_2 - a_2 b_1 \rangle$$

Ex. Compute $\vec{a} \times \vec{b}$ if

$$\vec{a} = \langle 2, 1, -2 \rangle \text{ and } \vec{b} = \langle 3, 2, 4 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \vec{k}$$

$$= (4+4)\vec{i} - (8+6)\vec{j} + (4-3)\vec{k}$$

$$= 8\vec{i} - 14\vec{j} + \vec{k}$$

OR $\langle 8, -14, 1 \rangle$

Show $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= - \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= - \vec{b} \times \vec{a}$$

Important fact:

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$

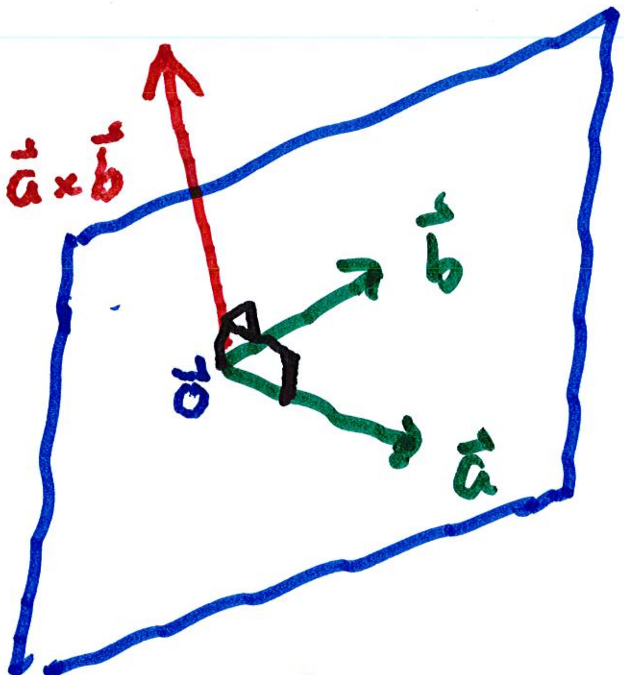
$$(\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$= (a_2 b_3 - a_3 b_2) a_1 + (a_3 b_1 - a_1 b_3) a_2 + (a_1 b_2 - a_2 b_1) a_3$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0$$

To show $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

is the same.



$\vec{a} \times \vec{b}$ is $\perp$ to the plane

containing $\vec{a}$ and $\vec{b}$

Ex. Find a vector of length 2 that is $\perp$ to both $\langle 2, 1, -2\rangle$ and $\langle -1, 1, 2\rangle$.

Compute $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ -1 & 1 & 2 \end{vmatrix}$

$= (2+2)\vec{i} - (4-2)\vec{j} + (2+1)\vec{k}$

$= 4\vec{i} - 2\vec{j} + 3\vec{k} \qquad (= \vec{w})$

$$|\vec{w}| = \sqrt{16+4+9} = \sqrt{29}$$

$$\vec{u} = \left\langle \frac{4}{\sqrt{29}}, \frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}} \right\rangle$$

$\vec{u}$ is a unit vector

To get $\vec{v}$, multiply by 2

$$\vec{v} = \left\langle \frac{8}{\sqrt{29}}, \frac{-4}{\sqrt{29}}, \frac{6}{\sqrt{29}} \right\rangle$$

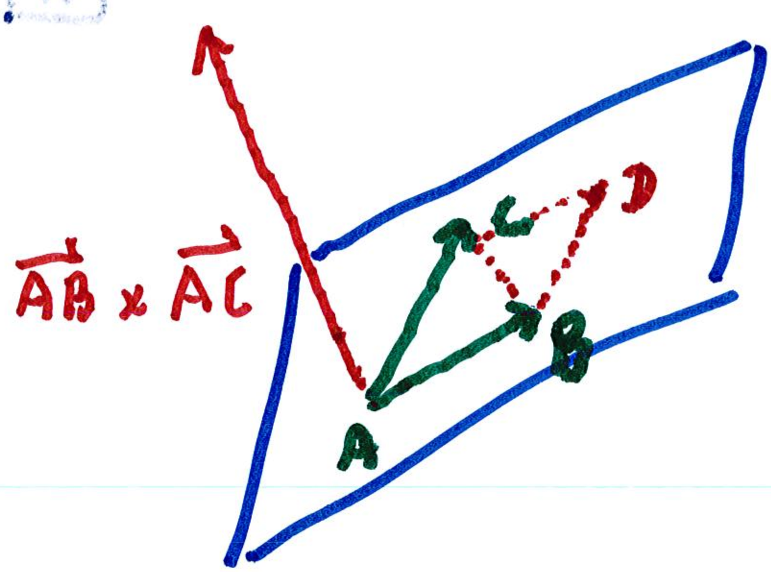
$\therefore$ answer is

$$\vec{v} = \left\langle \frac{8}{\sqrt{29}}, \frac{-4}{\sqrt{29}}, \frac{6}{\sqrt{29}} \right\rangle$$

Ex. Find a vector $\perp$ to the plane containing $A(1,0,1)$, $B(1,2,4)$ and $C(3,1,2)$

$\overrightarrow{AB} = \langle 0, 2, 3 \rangle$ and

$\overrightarrow{AC} = \langle 2, 1, 1 \rangle$



A

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= -\vec{i} - (-6)\vec{j} + (-4)\vec{k}$$

$$= -\vec{i} + 6\vec{j} - 4\vec{k}$$

What is the size of $\vec{a} \times \vec{b}$,

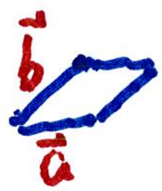
i.e., what is $|\vec{a} \times \vec{b}|$?

$$\boxed{|\vec{a} \times \vec{b}| = \text{Area of parallelogram generated by } \vec{a} \text{ and } \vec{b}.}$$

Ex. What is the area of the parallelogram with verticies A, B, C, D?

As above, set $\vec{a} = \overrightarrow{AB}$

and $\vec{b} = \overrightarrow{AC}$.

Area of 

$= \left| \vec{a} \times \vec{b} \right|$

$= \left| -\vec{i} + 6\vec{j} - 4\vec{k} \right|$

$= \sqrt{1 + 36 + 16} = \sqrt{53}$.

The area of the triangle

$ABC = \frac{1}{2}\sqrt{53}$

Also, if $\vec{a} \neq 0$ and $\vec{b} \neq 0$,

and not multiples of each other,

then the area of the

parallelogram generated

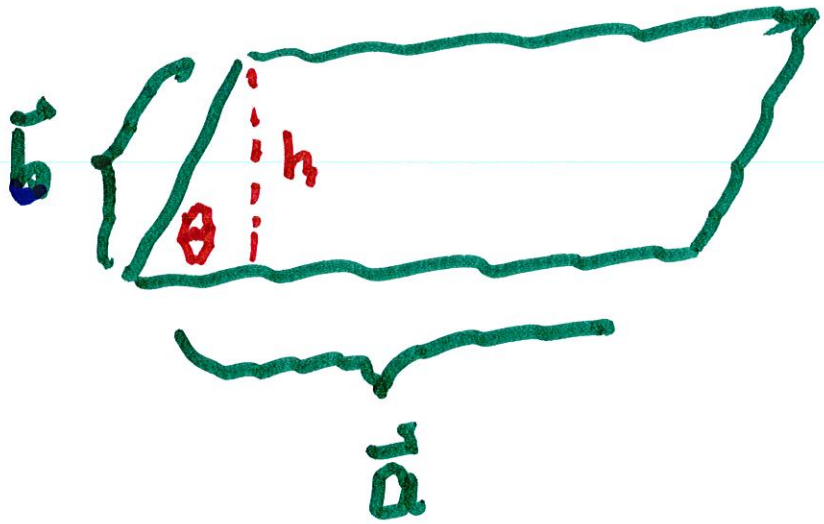
by $\vec{a}$ and $\vec{b}$ is

$0 < \theta < \pi$

$$A = |\vec{a}||\vec{b}| \sin\theta \quad . \quad (1)$$

$$\boxed{\begin{aligned} |\vec{a} \times \vec{b}| &= \text{Area of parallelogram} \\ &= |\vec{a}||b| \sin\theta \end{aligned}}$$

To see that (1) is true,



Note $\dfrac{h}{|\vec{b}|} = \sin\theta$, so $h = |\vec{b}|\sin\theta$

Area $= |\vec{a}|h = |\vec{a}||\vec{b}|\sin\theta$

10

$\boxed{15}$

Find $|\vec{v} \times \vec{w}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$= -2\vec{i} - (2)\vec{j} + 2\vec{k}$

$= -2\vec{i} - 2\vec{j} + 2\vec{k}$

$|\vec{v} \times \vec{w}| = \sqrt{4 + 4 + 4}$

$= \sqrt{12}$ = Area of 

Ex. Calculate the area of the parallelogram with vertices at $A(2,1,1)$, $B(3,1,2)$, $C(3,3,4)$ and $D(4,3,5)$

$$\overrightarrow{AB} = \langle 1, 0, 1 \rangle = \vec{v}$$

$$\overrightarrow{AC} = \langle 1, 2, 3 \rangle = \vec{w}$$

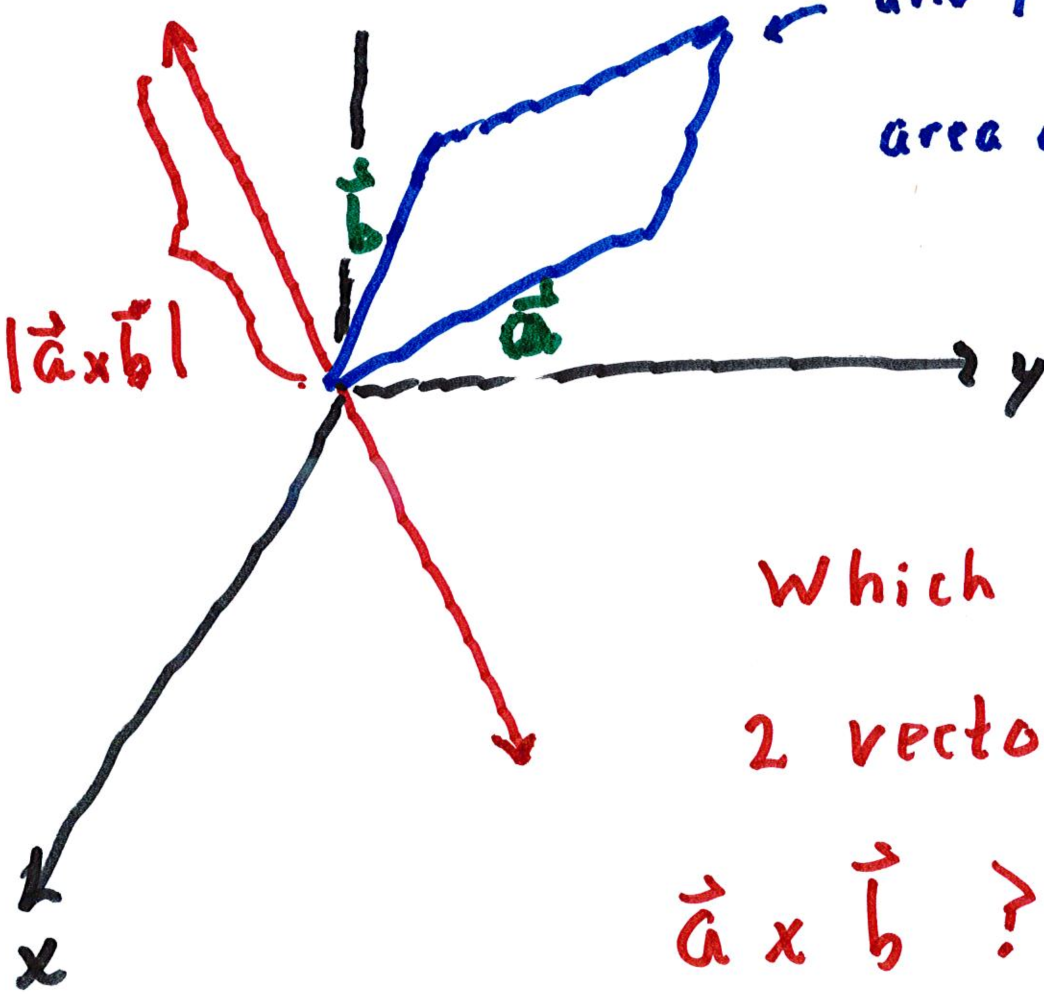
$$\overrightarrow{CD} = \langle 1, 0, 1 \rangle$$

Given 2 vectors $\vec{a}$ and $\vec{b}$,

we know $\vec{a} \times \vec{b}$ is $\perp$ to

$\vec{a}$ and $\vec{b}$

and $|\vec{a} \times \vec{b}| =$ area of Parallelogram



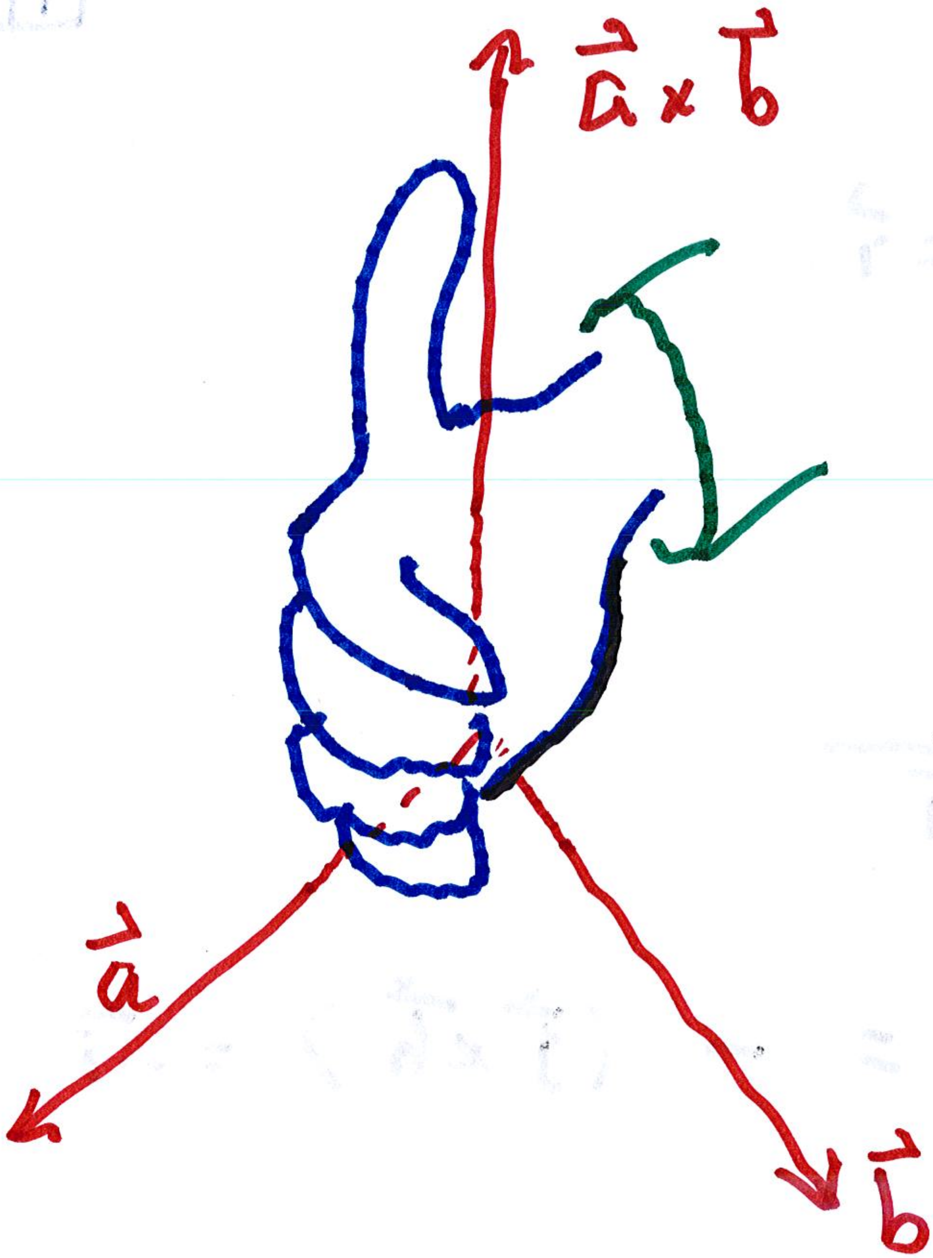
Which of the 2 vectors is $\vec{a} \times \vec{b}$?

Right-Hand Rule:

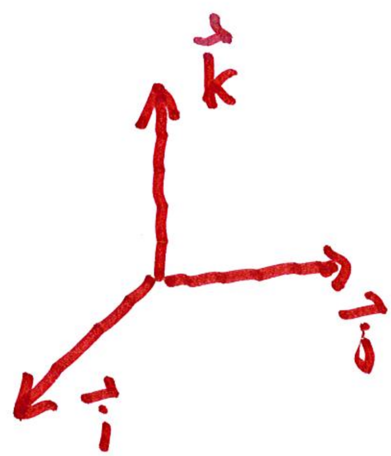
1. With your wrist at the origin, point your hand to $\vec{a}$.

2. Curl your fingers to $\vec{b}$.

3. Your thumb points to $\vec{a} \times \vec{b}$.

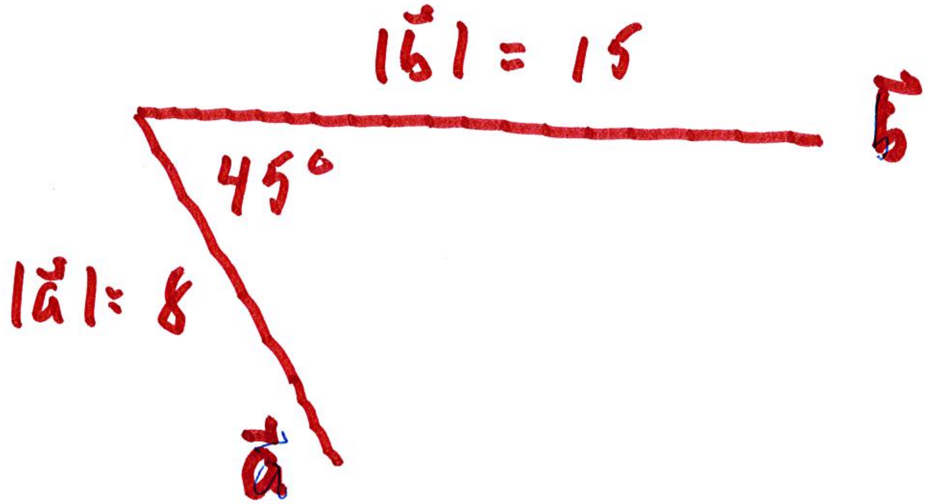


$\vec{i} \times \vec{j} = \vec{k}$



Ex. Suppose the angle from $\vec{a}$ to $\vec{b}$ is $= 45°$ and that $|\vec{a}| = 8$ and $|\vec{b}| = 15$.

~~what~~ what is $|\vec{a} \times \vec{b}|$? and is $\vec{a} \times \vec{b}$ directed into or out of the page?



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

$$= 8 \cdot 15 \cdot \frac{1}{\sqrt{2}}$$

$$= \underline{60\sqrt{2}}$$

$\vec{a} \times \vec{b}$ points out of page.

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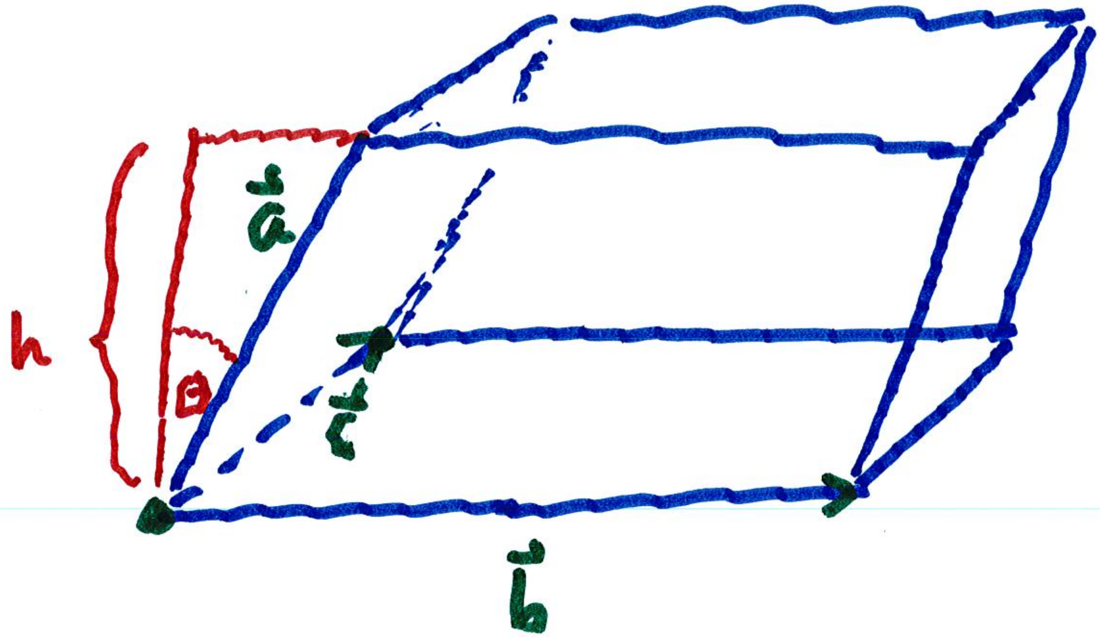
3 nonzero vectors

$\vec{a}$, $\vec{b}$, and $\vec{c}$



Note $|\vec{b} \times \vec{c}|$ = area of base

(the parallelogram)

$$\frac{h}{|\vec{a}|} = \cos\theta \rightarrow h = |\vec{a}|\cos\theta$$

$$\text{Vol.} = h \cdot \text{Area} = h\,|\vec{b} \times \vec{c}|$$

$$= |a|\cos\theta\;|\vec{b} \times \vec{c}|$$

$$= |\vec{a}||\vec{b} \times \vec{c}| \cos \theta$$

$$= \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

This is the <u>triple</u> <u>product</u>

The volume of the parallelo~~gram~~ piped

$$= \underline{\underline{\left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|}}$$

One can show that

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\therefore$ Vol of Par. generated by

$\vec{a}, \vec{b}, \vec{c}$ = abs. value of

det. of $\vec{a}$, $\vec{b}$, and $\vec{c}$

Ex. Find volume generated by

$\vec{a}$, $\vec{b}$, and $\vec{c}$, if

$\vec{a} = \langle -1, 2, 1 \rangle$, $\vec{b} = \langle 1, -3, 2 \rangle$

and $\vec{c} = \langle -1, 1, 2 \rangle$.

$$= \begin{vmatrix} -1 & 2 & 1 \\ 1 & -3 & 2 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= -1 \begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix}$$

$= 8 - 8 - 2$

$\therefore \text{Vol} = |-2| = \underline{\underline{2}}$

Ex. Given the force

$\vec{F} = \langle 4, 2 \rangle$, find the

projection of $\vec{F}$ in the

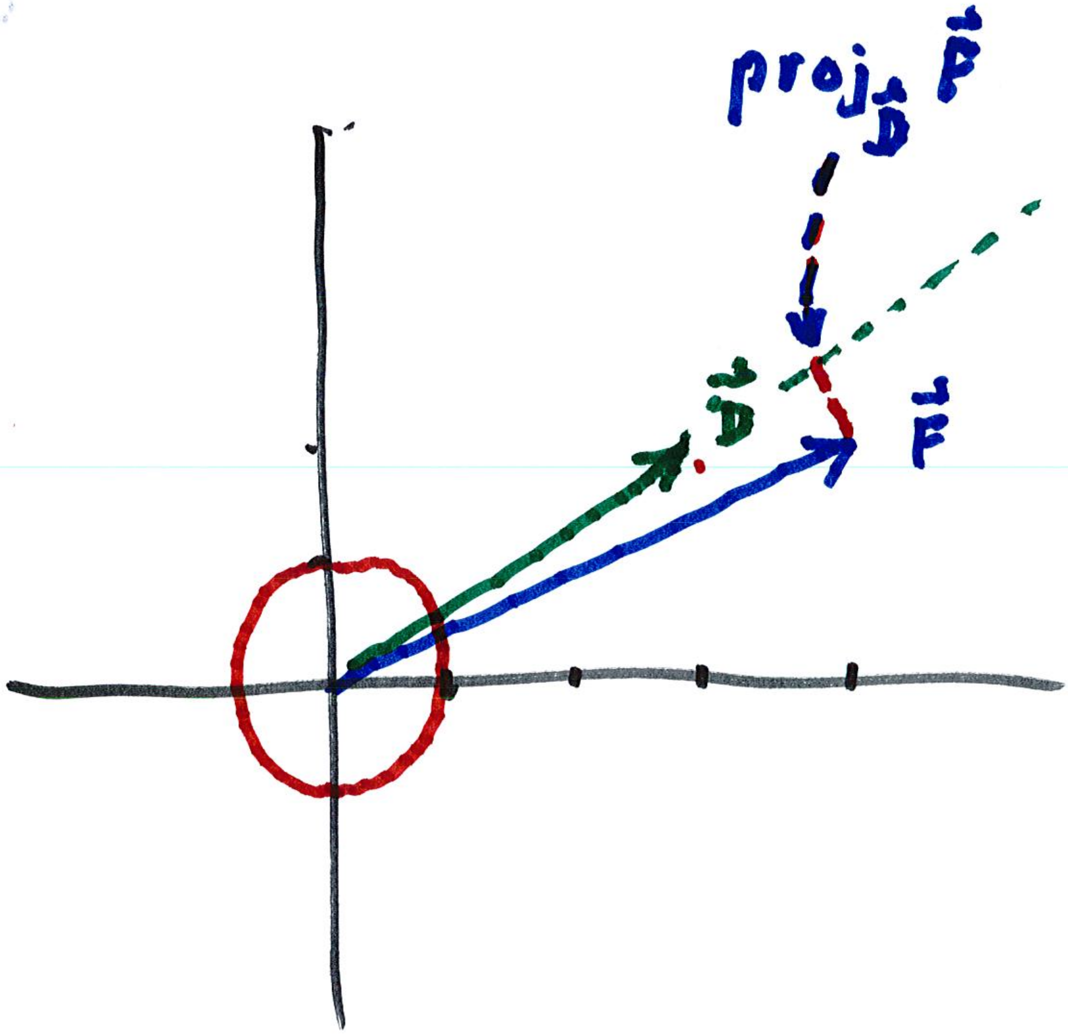
direction of $\vec{D} = \langle 3, 2 \rangle$.

$$\text{proj}_{\vec{D}}\vec{F} = \left(\frac{\vec{D}\cdot\vec{F}}{|\vec{D}|^2}\right)\vec{D}$$

$$= \frac{16}{13}\langle 3, 2\rangle = \left\langle \frac{48}{13}, \frac{32}{13}\right\rangle$$

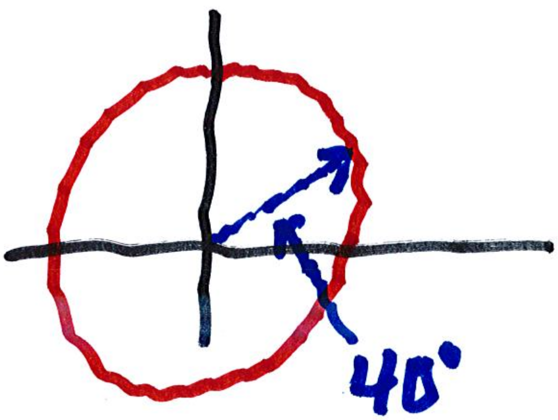
Find $\text{comp}_{\vec{D}}\vec{F} = \dfrac{\vec{D}\cdot\vec{F}}{|\vec{D}|}$

$$= \frac{16}{\sqrt{13}}$$



31. A man throws a football with an ~~elevation~~ angle of elevation 40° and speed

30 ft/s. Find the horizontal and vertical components of the velocity vector.



$$\vec{u} = (\cos 40)\vec{i} + (\sin 40)\vec{j}$$

$$\vec{v} = 30(\cos 40)\vec{i} + 30(\sin 40)\vec{j}$$

$\boxed{30}$ ~~25~~

$\therefore v_{hor} = 30(\cos 40)$

and $v_{vert} = 30(\sin 40)$

Ex. Find angle between

$\langle 4,0,2 \rangle$ and $\langle 2,-1,0 \rangle$

$\vec{a}$ $\qquad\qquad$ $\vec{b}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{8}{\sqrt{20}\sqrt{5}}$$

$$= \frac{8}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$