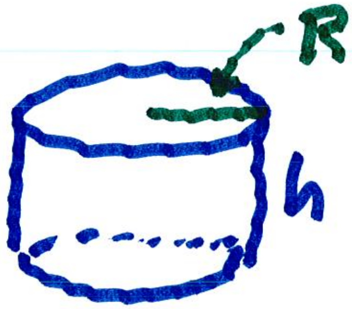
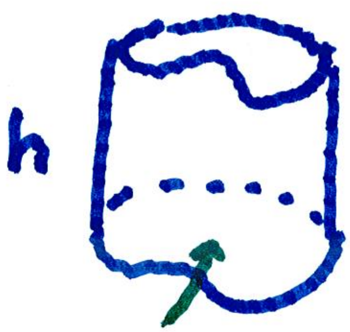
6.2 Volumes

Volume of cylinder is $V = \pi R^2 h$



More generally,

a generalized cylinder

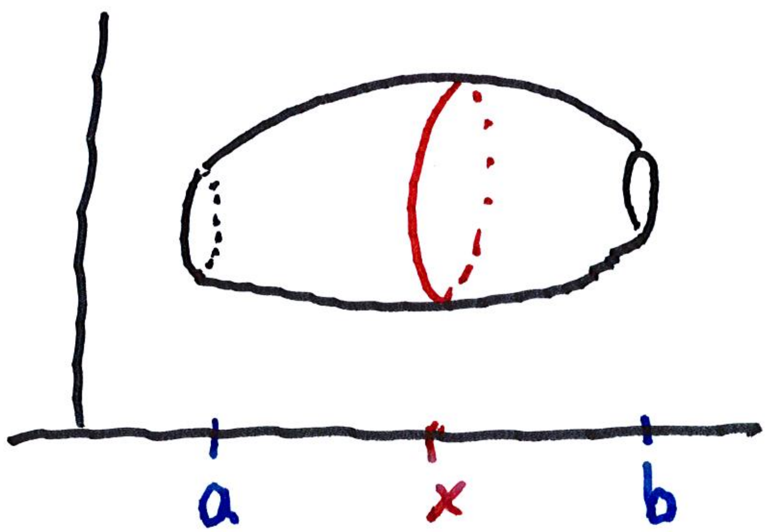


Then $V = Ah$

B = base

A = area of base

Now consider a region like an unsliced loaf of bread:



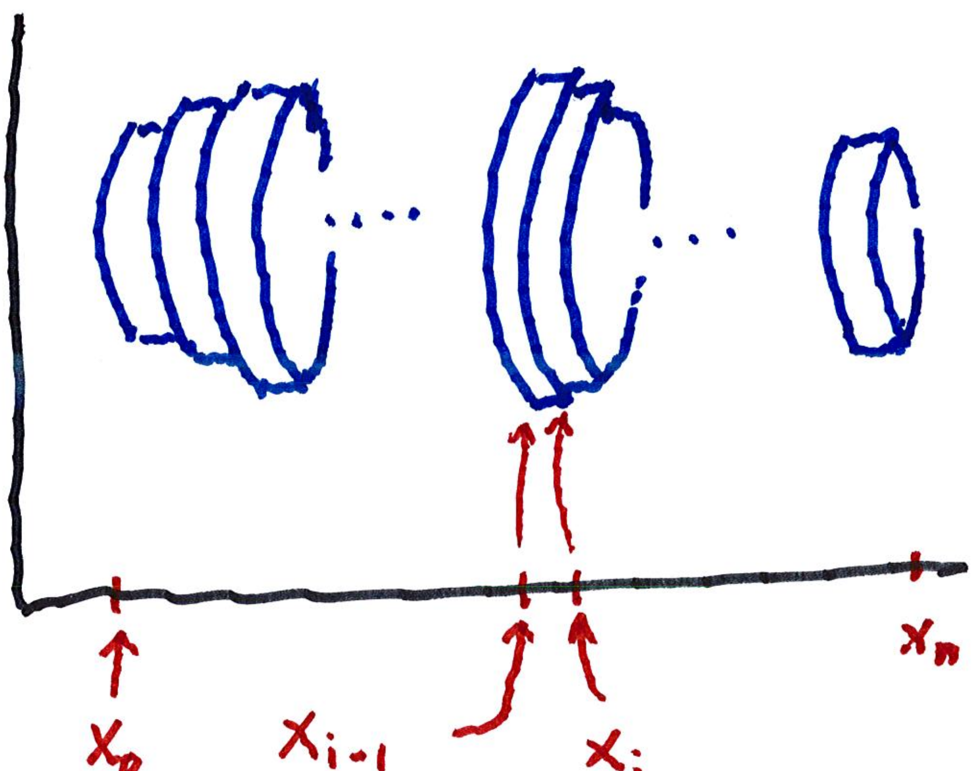
$A(x) =$ area of x-cross section

where $a \le x \le b$.

Now set $\Delta x = \dfrac{b-a}{n}$.

with

$$a = x_0 < x_1 \cdots < x_{i-1} < x_i < \ldots x_n = b$$



Volume of slice between

x_{i-1} and x_i is $\approx A(x_i)\,\Delta x$

Volume of all n slices is

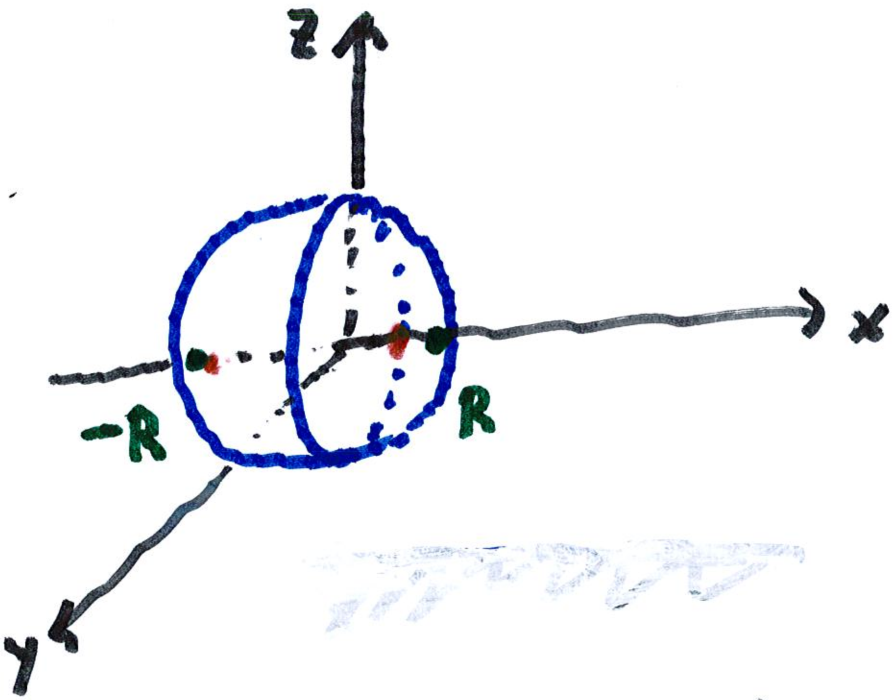
$$\approx \quad V = \sum_{i=1}^{n} A(x_i)\,\Delta x$$

As $n \to \infty$

$$V = \int_a^b A(x)\,dx$$

Ex. Find volume of sphere of radius R

$$x^2 + y^2 + z^2 = R^2$$



The x-slice of sphere is

$$y^2 + z^2 = R^2 - x^2$$

The radius $= \sqrt{R^2 - x^2}$

$$\therefore \quad A(x) = \pi \left(\sqrt{R^2 - x^2} \right)^2$$

$$A(x) = \pi \left(R^2 - x^2 \right)$$

$$\text{So } V = \int_{-R}^{R} \pi \left(R^2 - x^2 \right) dx$$

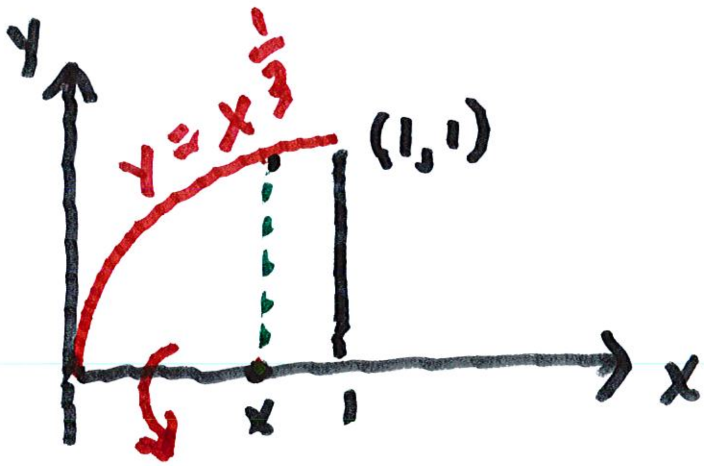
even

$$= 2\pi \int_{0}^{R} \left(R^2 - x^2 \right) dx$$

$$= 2\pi \left(R^2 x - \frac{x^3}{3} \right) \Bigg|_0^R$$

$$= 2\pi \left(R^3 - \frac{R^3}{3} \right) = \frac{4\pi R^3}{3}$$

Ex. Consider the region bounded by $y = x^{1/3}$, $y = 0$, and $x = 1$. What is volume of solid obtained by revolving it around x-axis?



The dotted line generates

a disk of radius $R_x = x^{1/3}$.

$\therefore A(x) = \pi x^{2/3}$

$$\text{Vol} = \int_0^1 \pi x^{2/3}\,dx = \pi \cdot \frac{3}{5} x^{5/3} \Big|_0^1$$

$$= \frac{3\pi}{5}$$

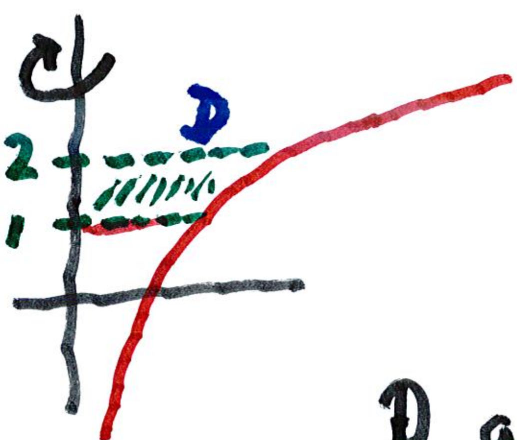
Sometimes, we use $A(y)$

= area of y-cross-section.

Ex. Let D = region bounded

by $y = \ln x$, $y = 1$, $y = 2$

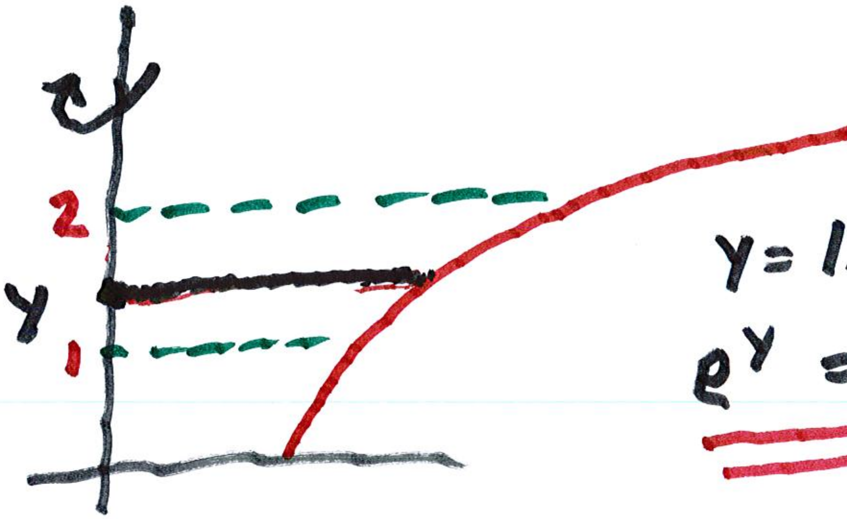
and $x = 0$



What is volume

obtained by rotating

D around <u>y-axis</u> ?



The segment at y generates a disk of radius $x = e^y$.

$\therefore A(y) = \pi x^2 = \pi (e^y)^2 = \pi e^{2y}$

So, $V = \int_1^2 \pi e^{2y}\, dy$

$= \frac{\pi}{2} e^{2y} \Big|_1^2 = \frac{\pi}{2}\left(e^4 - e^2\right)$

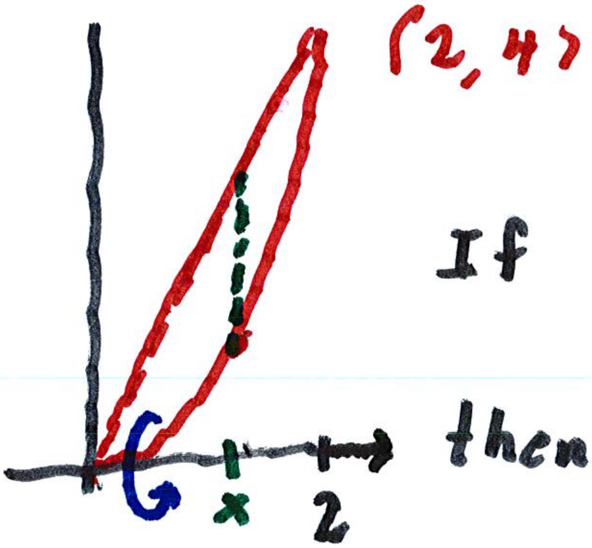
So far we've used the disk method. Now we use the <u>washer</u> <u>method</u>.

Ex. Find the volume of region obtained by revolving

D, region bounded by

$y = 2x$ and $y = x^2$

about the x-axis.



If $x^2 = 2x$,

then $x = 0$ or $x = 2$

The dotted segment generates a ring. The inside radius $= x^2$

and outer radius $= 2x$

Area of ring

is $\pi(2x)^2 - \pi(x^2)^2 = A(x)$

or $A(x) = \pi(4x^2 - x^4)$

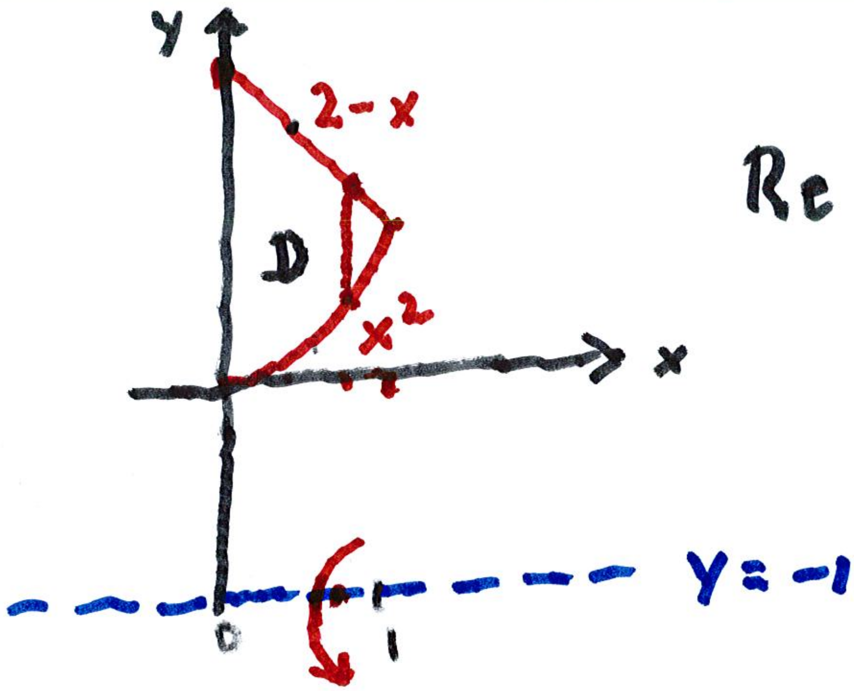
$$\text{Vol} = \int_0^2 \pi(4x^2 - x^4)\,dx$$

$$= \pi\left(\frac{4x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}$$

Ex Let D be bounded by

$y = 2 - x$, $y = x^2$, and $x = 0$



Re

Now revuolve D about $y = -1$

Find volume.

The outer radius (for fixed x)

is $(2-x)-(-1) = 3-x$.

The inner radius (for fixed x)

is $x^2-(-1) = x^2+1$.

Note $x^2 = 2-x \rightarrow x^2+x-2=0$

or $(x+2)(x-1)=0$

$\rightarrow x=1$

$$\text{Vol} = \int_0^1 \pi(3-x)^2 - \pi(x^2+1)^2\,dx$$

$$= \pi \int_0^1 -x^4 + x^2 - 2x^2 - 6x + 8 \, dx$$

$$= \pi \left(-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right|_0^1$$

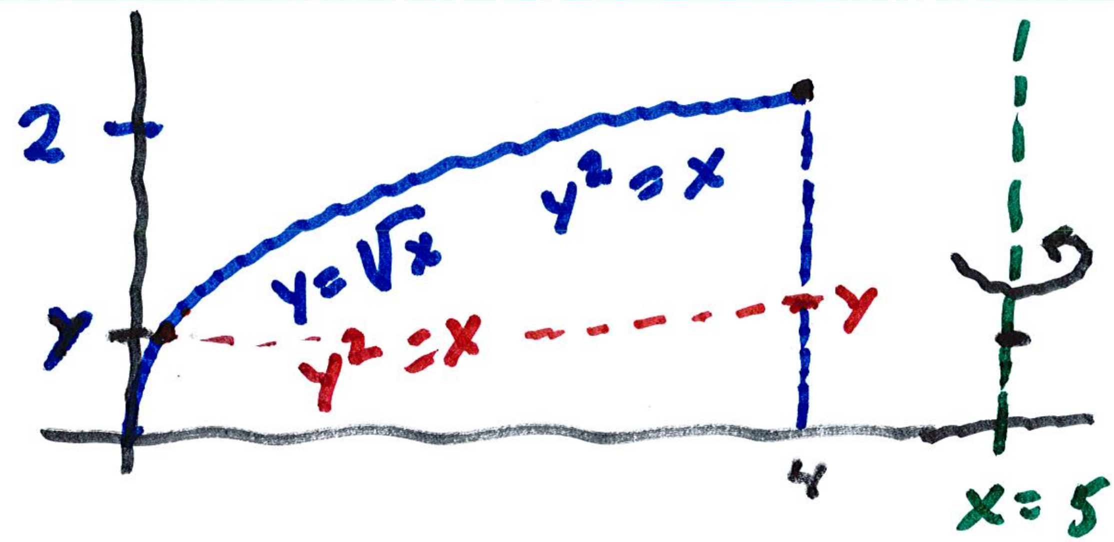
$$= \pi \left(-\frac{1}{5} - \frac{1}{3} + 5 \right) = \frac{67\pi}{15}$$

Ex. Let D in $\mathbb{R}^2$ be bounded

by $y = \sqrt{x}$, $y = 0$, and $x = 4$

What is volume if we revolve

D about the line $x = 5$?



Outer radius for fixed y is

o.r. $= 5 - y^2$ and

i.r. $= 1$

inner radius

$$\therefore \text{Vol} = \int_0^2 \pi\,(5-y^2)^2 - \pi \cdot 1^2 \, dy$$

$$= \pi \int_0^2 (24 - 10y^2 + y^4)\, dy$$

$$= \pi \left(24y - \frac{10y^3}{3} + \frac{y^5}{5} \right) \Big|_0^2$$

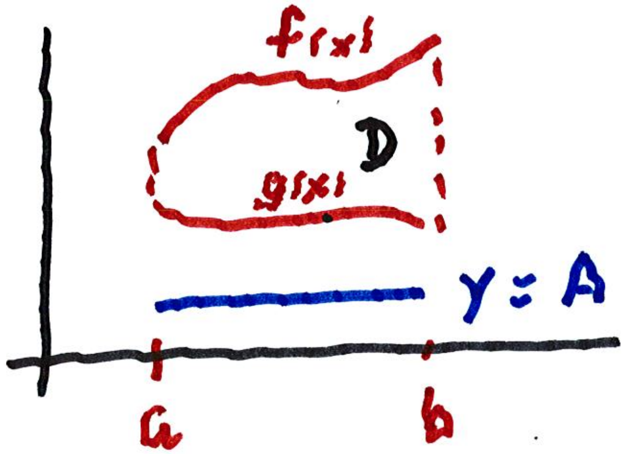
$$= \pi \left(48 - \frac{80}{3} + \frac{32}{5} \right)$$

$$= \frac{316}{15} \pi$$

Now let D be the region

$a \le x \le b \quad , \quad g(x) \le y \le f(x)$

Suppose we rotate D about the line $y = A$. Find the volume



The inner radius is

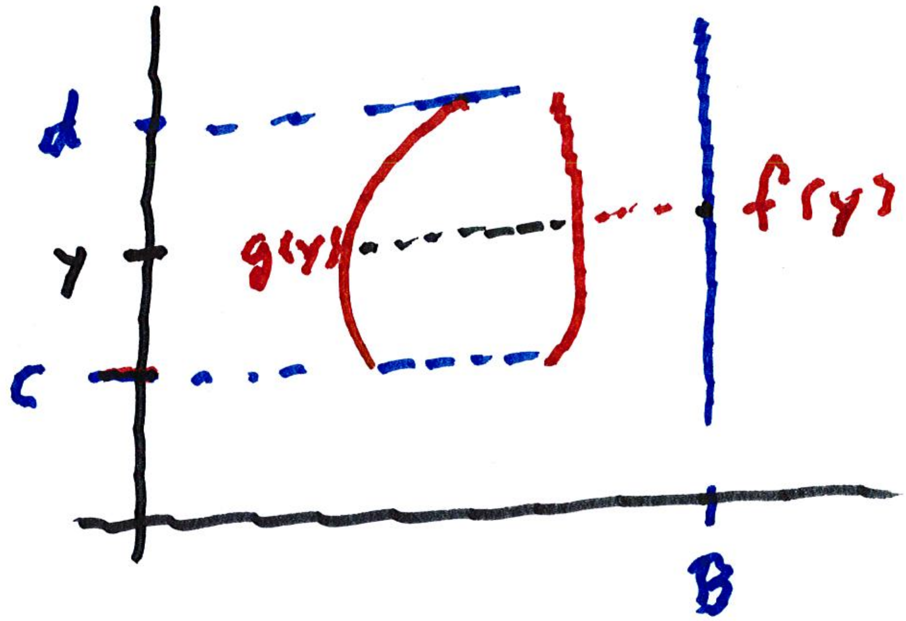
In. Rad $= g(x) - A$

The outer radius is

O. Rad $= f(x) - A$

$$V = \int_a^b \pi \left(f(x) - A\right)^2 - \pi \left(g(x) - A\right)^2 dx$$

Suppose D is $\begin{cases} c \le y \le d \\ g(y) \le x \le f(y) \end{cases}$



Rotate D about $x = B$

I.R. $= B - f(y)$

O.R. $= B - g(y)$

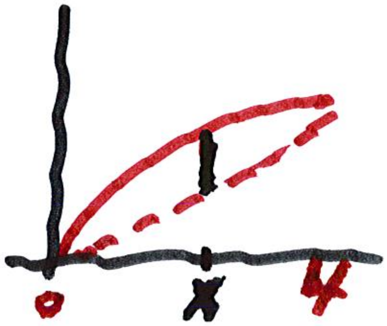
$$V = \pi \int_c^d (B - g(y))^2 - (B - f(y))^2 \, dy$$

Ex. The base of a solid S is the region bounded by

$y = \sqrt{x}$ and $y = \frac{x}{2}$.

Cross sections perpendicular to the x-axis are squares.

Find the volume of S

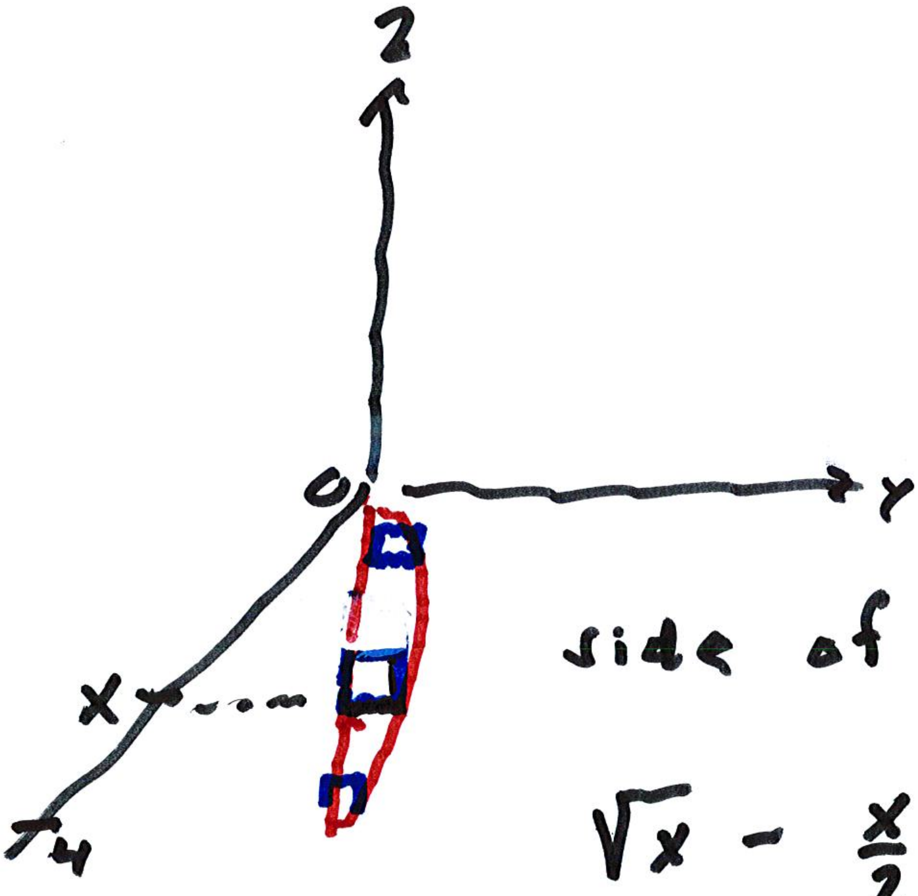


$\sqrt{x} = \frac{x}{2}$

$x = \frac{x^2}{4}$

$4x = x^2$

$\rightarrow x = 0$ or $x = 4$.



side of square is

$\sqrt{x} - \frac{x}{2}$

$$\text{Vol.} = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx$$

$$= \int_0^4 x - x^{3/2} + \frac{x^2}{4} \, dx$$

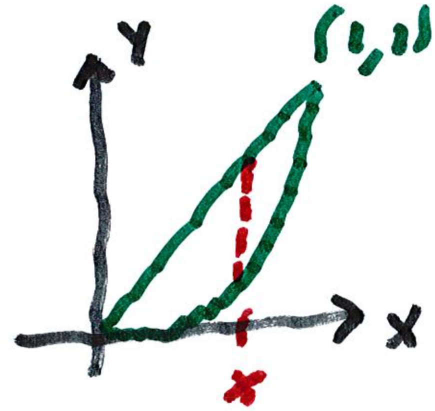
$$= \frac{x^2}{2} - \frac{2}{5} x^{5/2} + \frac{x^3}{12} \Big|_0^4$$

$$= 8 - \frac{64}{5} + \frac{16}{3}$$

$$= \frac{8}{15}$$

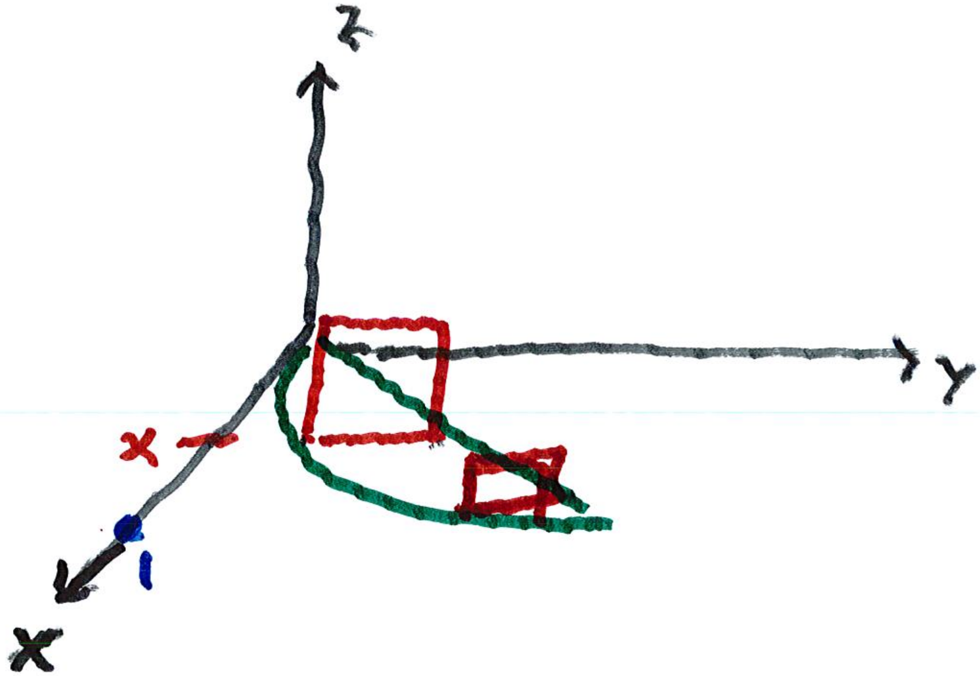
Other solids:

Ex. Let D = region in plane $z=0$ that is bounded by $y=x$ and $y=x^2$



Suppose S is a solid with base D such that each cross-section perpendicular to the x-axis is a square.

24 50



side of square is $(x - x^2)$

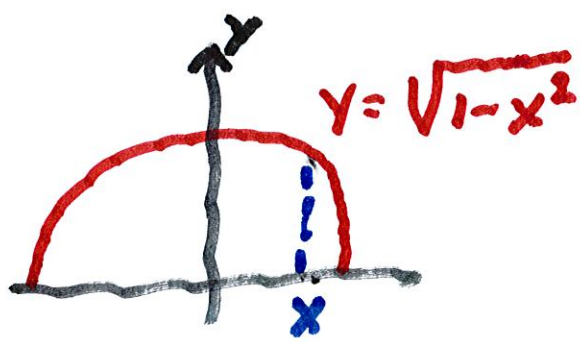
$$\therefore \quad A(x) = (x - x^2)^2$$

$$\Rightarrow \text{Vol} = \int_0^1 (x - x^2)^2 \, dx$$

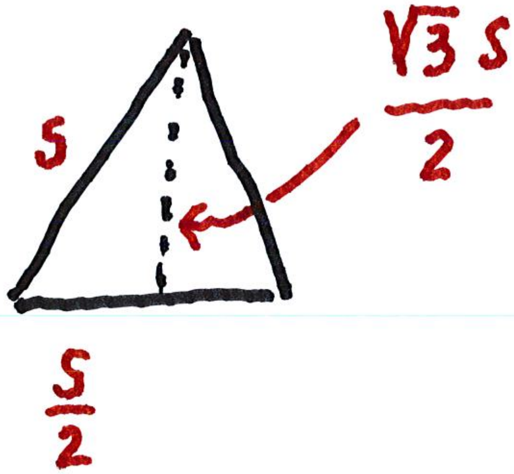
$$= \left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right) \Bigg|_0^1 =$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$$

Now suppose D is bounded by $y = \sqrt{1-x^2}$ and $y = 0$,



and that the x-cross-sections $\perp$ to base D are equilateral triangles.



Area of Equi-Δ is $A = \frac{\sqrt{3}\,s}{4}$

$\therefore$ If $s = \sqrt{1-x^2}$, then

$$A(x) = \frac{\sqrt{3}\,(1-x^2)}{4}$$

$$\therefore V = \int_{-1}^{1} \frac{\sqrt{3}\,(1-x^2)}{4}\,dx = \frac{\sqrt{3}}{3}$$